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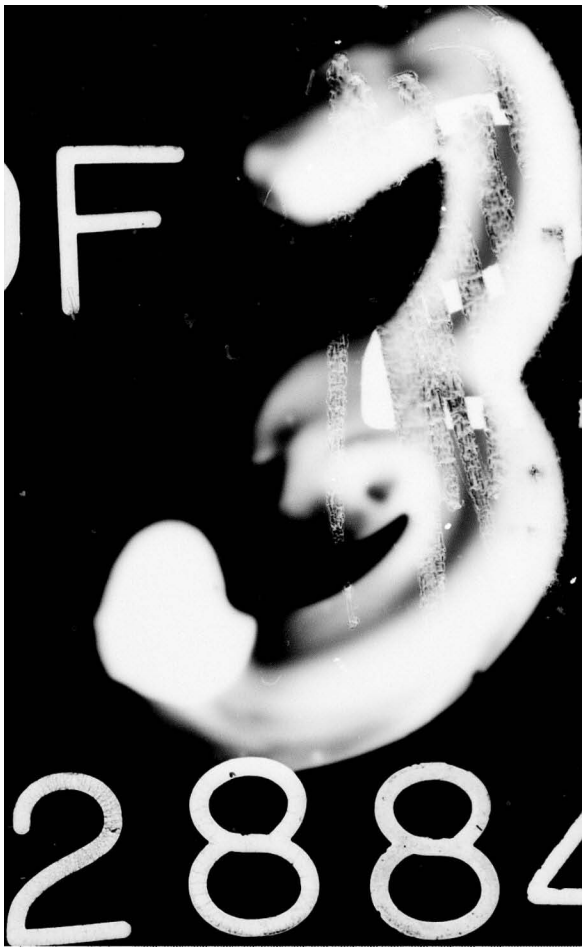
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LEVEL II

Generating Covariance Sequences and the Calculation of Quantization and Rounding Error Variances in Digital Filters

by

A. A. Beex

ONR Technical Report #26

July 1979

Prepared for the Office of Naval Research
under Contract N00014-75-C-0518

L. L. Scharf, Principal Investigator

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Modeling and estimation procedures for covariance sequence and spectrum approximation are developed in this thesis. The covariance sequence is modeled as a complex linear combination of damped complex exponentials. This model arises naturally as the representation for the covariance sequence associated with a strictly proper ARMA(M,N) system driven by white noise. Related to this seemingly natural covariance model is a synthesis procedure for a subclass of wide sense stationary ARMA(M,N) processes. The resulting spectral representation for the covariance sequence is a positive real linear

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20. combination of damped complex exponentials, a generalization of the standard representation in terms of stochastic almost periodic functions. The importance of the generalized structure lies in the more efficient representation of a large class of wide sense stationary processes. For this parametric approach estimation techniques are developed that lie in philosophy somewhere between the nonlinear least squares approach and the tractable modified least squares procedure. The resulting parameter estimation equations are linear, except for a single polynomial rootfinding problem that must be solved.

Parametric spectra are efficiently approximated for a large class of processes using the covariance sequence approximant. The generality of the approximant does not force the sequence to be non-negative definite, so that the corresponding "spectrum" estimate can be negative at some frequencies. The same generality of the model permits an accurate representation of sequences that are not non-negative definite.

Recursive digital filter design based on second order statistical information is achieved by covariance sequence approximation. The stochastic minimal realization problem encountered in covariance invariant recursive digital filter design forms a special case, and results in stable ARMA(M,M-1) filters with real coefficients. For a general desired spectrum the resulting filter is stable and of ARMA(2M,2M-2) type with real coefficients. The latter design is efficiently achieved by approximating the covariance sequence associated with the square root of the desired spectrum. The phase characteristic associated with these designs can be manipulated to some extent to achieve linear phase, phase equalization, or filter invertibility properties.

Covariance Sequence Approximation with Applications to Spectrum Analysis and Digital Filter Design[1]

by

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[1]This work partially supported by the Office of Naval Research, Statistics and Probability Branch, Arlington, VA, under Contract N00014-75-C-0518.

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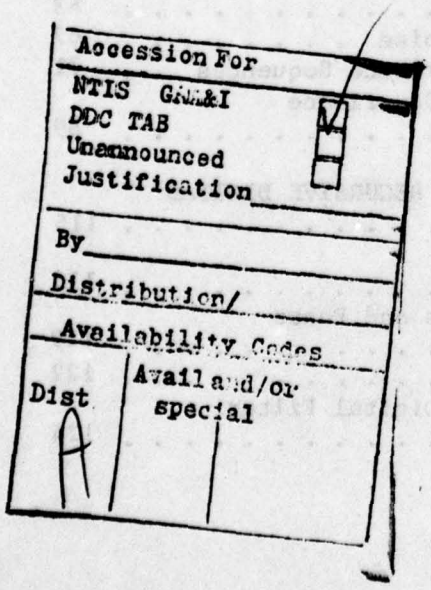

Abstract

Covariance Sequence Approximation with Applications to Spectrum Analysis and Digital Filter Design

Modeling and estimation procedures for covariance sequence and spectrum approximation are developed in this thesis. The covariance sequence is modeled as a complex linear combination of damped complex exponentials. This model arises naturally as the representation for the covariance sequence associated with a strictly proper ARMA(M,N) system driven by white noise. Related to this seemingly natural covariance model is a synthesis procedure for a subclass of wide sense stationary ARMA(M,N) processes. The resulting spectral representation for the covariance sequence is a positive real linear combination of damped complex exponentials, a generalization of the standard representation in terms of stochastic almost periodic functions. The importance of the generalized structure lies in the more efficient representation of a large class of wide sense stationary processes. For this parametric approach estimation techniques are developed that lie in philosophy somewhere between the nonlinear least squares approach and the tractable modified least squares procedure. The resulting parameter estimation equations are linear, except for a single polynomial root-finding problem that must be solved.

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1 INTRODUCTION

The theory of parametric statistical inference is concerned with the identification of parametric models from random data. When the data arises in the form of a time series, then the basic problem is to identify a parametric model for the underlying stochastic process. If the underlying process is normal, then the covariance sequence provides a complete statistical characterization of the process. Furthermore, the covariance sequence for a process lies in one-to-one correspondence with the power spectral density and summarizes all the information required to realize (or approximate) a minimum mean-squared error (MMSE) filter for the process. If the power spectral density happens to be rational, then the MMSE filter is of Wiener/Kalman variety.

All of this suggests that one might profitably describe a time series by identifying a parametric model for its covariance sequence. In this thesis we develop techniques for identifying parameters in the following parametric model of the covariance sequence:

$$R_k = \sum_i A_i \, p_i^{|k|} \qquad 1.1$$

This model is a useful one because it includes the covariance sequences of the following processes as special cases: (i) autoregressive moving average (ARMA), (ii) random amplitude, random phase sinusoid, (iii) white noise, and (iv) linear combinations of (i) - (iii). Furthermore, the model reduces under special circumstances to the model implicit in DFT analysis of random data and in Pisarenko's decomposition of a covariance sequence.

Once the parameters $\{A_i\}$, $\{p_i\}$ of (1.1) have been determined, it is a simple matter to find the corresponding parametric spectral density estimate. Applying the covariance sequence model of (1.1) to the covariance sequence associated with the square root of a desired spectral density, one may design recursive digital filters with a magnitude squared frequency response that approximates the desired spectral density.

In Chapter 2 spectral representations are given for stochastic processes and covariance functions. The relations between the proposed covariance sequence approximant and the DFT and Pisarenko decomposition are explored. A generalized stochastic process synthesis equation is developed for a subset of those processes whose covariance sequences are positive real linear combinations of complex exponentials.

In Chapter 3 we develop a modified least squares fit of the covariance sequence approximant of (1.1) to a given covariance sequence. A decoupling of the minimization problem for the complex exponentials and the complex weights of the linear combination results. An efficient recursive procedure is outlined for the determination of a minimum phase covariance sequence predictor polynomial.

In Chapter 4 the parametric spectral estimate is derived that is associated with the covariance sequence approximant of this thesis. We show that harmonic processes in white noise can be represented as a uniform limit of this spectral estimate. The covariance sequence approximant is then applied to finite, known covariance sequences as well as finite, estimated covariance sequences for harmonic processes in noise. As a test of the robustness of the covariance sequence approximant we apply it to a Gaussian covariance sequence.

By Parseval's theorem we can interpret covariance sequence approximation as spectral density approximation, so that covariance sequence approximation seems the natural approach to designing recursive digital filters from magnitude squared frequency responses. An efficient design procedure is developed in Chapter 5. Design examples are given for some representative lowpass filter designs of ARMA(2M,2M$^-$) type. Important features are the stability of the filter and the efficiency of the design algorithm.

In Chapter 6 conclusions are advanced and recommendations for extensions are proposed.

2 SPECTRAL REPRESENTATIONS FOR STOCHASTIC PROCESSES AND COVARIANCE FUNCTIONS

An introductory exposition is given in Section 2.1 for general spectral representations of wide sense stationary stochastic processes and their corresponding covariance functions [K], [Y], [F], [BR].

In the following section the spectral representation for the covariance function of a linear filter driven by white noise is treated. The main result of this section is that for strictly proper ARMA(M,N) systems the spectral representation for the covariance function can always be written as a complex linear combination of complex damped exponentials.

Section 2.3 gives the spectral representation for the stochastic process and a stochastic process synthesis equation for the spectral analysis procedure based on the Discrete Fourier Transform.

The spectral representation for the stochastic process and the corresponding stochastic process synthesis equation underlying the Pisarenko decomposition spectral analysis procedure are treated in Section 2.4.

The final section of this chapter extends the synthesis structure of Sections 2.3 and 2.4 to allow the generation of wide sense stationary processes for which the covariance function is represented by a positive real linear combination of complex damped exponentials. The latter is important since it provides a finite parameter representation for a large class of wide sense stationary stochastic processes.

2.1 Spectral Representations for Weakly Stationary Processes

A weakly stationary or wide sense stationary process $\{X(t), t \in R\}$ satisfies the following moment conditions:

By Parseval's theorem we can interpret covariance sequence approximation as spectral density approximation, so that covariance sequence approximation seems the natural approach to designing recursive digital filters from magnitude squared frequency responses. An efficient design procedure is developed in Chapter 5. Design examples are given for some representative lowpass filter designs of ARMA(2M,2M$^-$) type. Important features are the stability of the filter and the efficiency of the design algorithm.

In Chapter 6 conclusions are advanced and recommendations for extensions are proposed.

$$E\,X(t) = E\,X(0) = m \;;\; -\infty < t < \infty \qquad 2.1a$$

and

$$E\,X(t+\tau)X(t) = E\,X(\tau)X(0) = C_X(\tau) \qquad 2.1b$$

We will now state the spectral representation theorem for wide sense stationary processes.

Spectral Representation Theorem:

Let $\{X(t),\ t\epsilon R\}$, be a real-valued weakly stationary process with $E\,X(t) = 0$ and covariance function $C_X(\tau) = E\,X(t+\tau)X(t)$. We further assume that $C_X(\tau)$ is continuous at $\tau=0$, in which case $C_X(\tau)$ is continuous for all τ. On the basis of these assumptions there exists a complex-valued random set function $Z(A)$, called the random spectral measure of the process, such that the process has the spectral representation

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, Z(d\lambda) \qquad 2.2$$

Here the integral is a Riemann-Stieltjes integral and the sense of the equality is that the Riemann-Stieltjes sum converges as a limit in the mean to $X(t)$.

The random spectral measure has the following properties

$$Z(-A) = Z^*(A) \qquad 2.3$$

which is a consequence of assuming $X(t)$ to be real-valued. It follows that

$$E\,Z(A) = 0 \qquad 2.4$$

since $E\,X(t) = 0$. If $F(A)$ is defined by $F(A) = E|Z(A)|^2$, then $F(A)$ is a measure called spectral distribution and the following relationship is valid:

$$E\,Z(A)Z^*(B) = F(A \cap B) \qquad 2.5$$

Note that if A and B are disjoint, $E\,Z(A)Z^*(B) = 0$, which says

that the random variables Z(A) and Z(B) are uncorrelated (or orthogonal). The latter two expressions can be put in the following operationally convenient form:

$$E\, Z(d\lambda)Z^*(d\mu) = \begin{cases} F_X(d\lambda) & \text{if } \mu = \lambda \\ 0 & \text{if } \mu \neq \lambda \end{cases} \qquad 2.6$$

where $F_X(d\lambda) = F((-\infty,\lambda+d\lambda]) - F((-\infty,\lambda])$. That is $d\mu$ and $d\lambda$ are thought of as being increments about μ and λ such that if $\mu \neq \lambda$, they are disjoint. For the covariance function of the real-valued wide sense stationary, zero mean process we then derive the spectral representation as follows:

$$\begin{aligned} C_X(\tau) &= E\, X(t+\tau)X(t) = \\ &= E\Big(\int_{-\infty}^{\infty} e^{i\lambda(t+\tau)} Z(d\lambda)\Big)\Big(\int_{-\infty}^{\infty} e^{i\mu t}\, Z(d\mu)\Big)^* \\ &= E\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\lambda(t+\tau)}\, e^{-i\mu t} Z(d\lambda)Z^*(d\mu) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\lambda-\mu)t} e^{i\lambda\tau} E\, Z(d\lambda)Z^*(d\mu) \\ &= \int_{-\infty}^{\infty} e^{i\lambda\tau} F_X(d\lambda) \end{aligned} \qquad 2.7$$

This result is known as the Bochner-Khintchine theorem.

Discrete time parameter, weakly stationary stochastic processes have a spectral theory which exactly parallels that given for continuous time processes. Let $\{X(k),\ k=0,\pm1,\ldots\}$ denote a weakly stationary process with discrete time parameter. Then the spectral representation of X(k) is

$$X(k) = \int_{-\pi}^{\pi} e^{i\lambda k} Z(d\lambda)\ ;\quad k=0,\pm1,\ldots \qquad 2.8$$

The spectral distribution function is given in terms of the spectral distribution F(A) by

$$F_X(\lambda) = F((-\pi,\lambda]) \tag{2.9}$$

The covariance sequence has the spectral representation

$$C_X(k) = E\, X(j+k)X(j)$$

$$= \int_{-\pi}^{\pi} e^{i\lambda k} F_X(d\lambda) \tag{2.10}$$

Note that the only real difference between continuous and discrete time parameter stochastic process representations is the interval of integration. With the Lebesgue decomposition of $F_X(A)$ we can give a more intuitive meaning to the spectral distribution function by expressing it as the sum of a discrete and a continuous component [C]:

$$F(A) = F^d(A) + F^c(A) \tag{2.11}$$

The continuous component can be further decomposed

$$F^c(A) = F^{ac}(A) + F^{sc}(A) \tag{2.12}$$

In the sequel we will neglect values on an ω set of probability zero [D], and leave the singular continuous component of $F^c(A)$ out of consideration. The discrete spectral distribution $F^d(A)$ is characterized by the spectral function $\{p_X(\lambda_j)\}$ indicating the spectral mass concentrated at frequency λ_j:

$$F^d(A) = \sum_{j:\lambda_j \in A} p_X(\lambda_j) \tag{2.13}$$

The continuous component $F^c(A)$ is determined by the derivative of the spectral distribution function, $f_X(\lambda) = \frac{d}{d\lambda} F^c(\lambda)$, where $f_X(\lambda)$ is called the spectral density function. Whenever the covariance function $C_X(\tau)$ is absolutely integrable or summable there is corresponding to it a continuous spectral density function $f_X(\lambda)$ [D].

In case the spectrum contains both discrete and continuous components the spectral mass or power in a set of frequencies A is:

$$F(A) = \sum_{j:\lambda_j \epsilon A} p_X(\lambda_j) + \int_A f_X(\lambda)d\lambda \qquad 2.14$$

For this mixed spectrum the following covariance function spectral representation is valid:

$$C_X(\tau) = \sum_{\lambda_j} e^{i\lambda_j \tau} p_X(\lambda_j) + \int e^{i\lambda\tau} f_X(\lambda)d\lambda \qquad 2.15$$

A similar decomposition of the random spectral measure exists

$$Z(A) = Z_d(A) + Z_c(A) \qquad 2.16$$

with

$$E\, Z_d(A) Z_c^*(B) = 0 \quad \forall\, A,B$$

Defining F(A) as before this leads precisely to the Lebesgue decomposition. Related to the Lebesgue decomposition is the following theorem due to Wold.

Wold Decomposition Theorem:

Let $\{X(t),\ t=0,\pm1,\ldots\}$, be a zero-mean, weakly stationary stochastic process. Then X(t) can be expressed as the sum of two zero-mean, weakly stationary processes,

$$X(t) = U(t) + V(t) \qquad 2.17$$

such that

i) the process U(t) is uncorrelated with the process V(t);

ii) U(t) has a one-sided moving average representation

$$U(t) = \sum_{k=0}^{\infty} a_k\, \xi(t-k) \text{ with } a_0=1 \text{ and } \sum_0^{\infty} a_k^2 < \infty\ ;\ \xi(t) \text{ is white;}$$

iii) the V(t) process is completely determined by linear functions of its past values.

The process V(t) is said to be deterministic and the process U(t) is

said to be nondeterministic. The spectral distribution of X(t) can be evaluated in terms of the spectral distributions of U(t) and V(t). With (2.17) and property i):

$$F_X(d\lambda) = F_U(d\lambda) + F_V(d\lambda) \qquad 2.18$$

The process U(t) has continuous spectral distribution due to property ii) above and therefore (2.18) is actually the Lebesgue decomposition of $F_X(\cdot)$. In all cases of practical interest then, the deterministic component is simply a stochastic almost periodic function:

$$V(t) = \sum_{j=-n}^{n} Z_j e^{i\lambda_j t} \quad ; \quad -\infty < t < \infty \qquad 2.19$$

where $\lambda_0, \lambda_{\pm 1}, \ldots, \lambda_{\pm n}$ are fixed frequencies and the Z_j are complex random variables. For V(t) to be real-valued and weakly stationary, we need the following properties:

$$\lambda_{-j} = \lambda_j \qquad j=0,\pm 1,\ldots,\pm n \qquad 2.20a$$

$$Z_{-j} = Z_j^* \qquad j=0,\pm 1,\ldots,\pm n \qquad 2.20b$$

$$E\, Z_j Z_k^* = 0 \quad \text{for } j \neq k \qquad 2.20c$$

The latter results in the following spectral representation for the covariance function of a stochastic almost periodic function

$$C_V(\tau) = \sum_{j=-n}^{n} E|Z_j|^2 e^{i\lambda_j \tau} \qquad 2.21$$

Using the properties (2.20a) and (2.20b) this can be written as

$$C_V(\tau) = \sum_{j=0}^{n} 2\sigma_j^2 \cos\lambda_j \tau \qquad 2.22$$

where $Z_j(\omega) = r_j(\omega) e^{j\phi_j(\omega)}$ in (2.19). This means that V(t) is a linear combination of sinusoids with random amplitudes $r_j(\omega) : (0,\sigma_j^2)$ and frequencies λ_j, each with a random phase term $\phi_j(\omega) : U[0,2\pi]$, where

$$E\, r_j(\omega) r_\ell(\omega) = \sigma_j^2\, \delta(j-\ell)$$
$$E\, \phi_j(\omega)\phi_\ell(\omega) = 0 \quad \forall\ j \neq \ell$$
$$E\, r_j(\omega)\phi_\ell(\omega) = 0 \quad \forall\ j,\ell \qquad 2.23$$

By virtue of the Wold decomposition theorem and the stochastic almost periodic function of (2.19) we can now consider the problem of analyzing sinusoids in noise via the spectral representation of weakly stationary stochastic processes.

2.2 Spectral Representation for Covariance Functions and Linear Operations

In this section we examine the effects of a linear filter on the spectral density function of a so-called white noise input. A white noise sequence is a sequence of uncorrelated, zero-mean random variables with common variance σ^2. This sequence is a weakly stationary process with covariance function

$$C(k) = \sigma^2 \delta(k) \quad ; \quad \delta(k) = \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases} \qquad 2.24$$

Since obviously $\{C(k)\} \varepsilon \ell_1$, there exists a continuous spectral density function, so that with (2.10)

$$C(k) = \int_{-\pi}^{\pi} e^{i\lambda k} f(\lambda) d\lambda \qquad 2.25$$

If $\{C(k)\} \varepsilon \ell_2$, as in most applications, then the numbers $C(k)/2\pi$ are the Fourier coefficients of the Fourier series expansion of the function $f(\lambda)$ periodically extended:

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-i\lambda k} C(k) \qquad 2.26$$

Equations (2.25) and (2.26) are the discrete time version of the Wiener-Khintchine relations. For the above defined white noise sequence, this readily yields

$$f(\lambda) = \sigma^2/2\pi \quad ; \quad |\lambda| \leq \pi \tag{2.27}$$

Similarly we can define a generalized weakly stationary process in the continuous time case by its covariance function

$$C(\tau) = \sigma^2\delta(\tau) \tag{2.28}$$

This function is not continuous at $\tau=0$, but has a continuous spectrum and spectral density

$$f(\lambda) = \sigma^2/2\pi \quad ; \quad -\infty < \lambda < \infty \tag{2.29}$$

Denote the output $Y(t)$ of the linear filter with transfer function $H(\lambda)$ excited with input $X(t)$ in the following operator notation

$$Y(t) = L\{X(t)\} \tag{2.30}$$

The action of a linear filter on $X(t)$ is completely determined by what it does to the complex exponential function $e^{i\lambda t}$ for all λ, which is determined by the transfer function $H(\lambda)$ of the filter. With the spectral representation theorem for $X(t)$, we can therefore write

$$\begin{aligned} Y(t) &= L\{X(t)\} \\ &= \int e^{i\lambda t} H(\lambda)\, Z_X(d\lambda) \end{aligned} \tag{2.31}$$

The latter represents the output of the linear filter as a weakly stationary stochastic process with spectral measure

$$Z_Y(d\lambda) = H(\lambda)\, Z_X(d\lambda) \tag{2.32}$$

Thus

$$\begin{aligned} F_Y(d\lambda) &= E\,|Z_Y(d\lambda)|^2 \\ &= |H(\lambda)|^2\, F_X(d\lambda) \end{aligned} \tag{2.33}$$

which indicates the following relations between the spectral functions

and spectral density functions at the input and output of the linear filter:

$$p_Y(\lambda) = |H(\lambda)|^2 \, p_X(\lambda) \tag{2.34a}$$

$$f_Y(\lambda) = |H(\lambda)|^2 \, f_X(\lambda) \tag{2.34b}$$

Let us now investigate the covariance sequence structure for some well known processes, resulting from white noise $(0,\sigma^2)$ driven AR, MA and ARMA systems respectively.

Assume given a stable AR system with transfer function

$$H_{AR}(z) = \frac{1}{1+a_1 z^{-1}+\ldots+a_M z^{-M}} \tag{2.35a}$$

$$= \frac{1}{A(z)} \tag{2.35b}$$

$$= \frac{1}{\prod_{i=1}^{M} (1-p_i z^{-1})} \tag{2.35c}$$

with real coefficients a_i and $|p_i| < 1$ (stability requirement). The spectral density function for the output then satisfies (2.34b) and (2.27):

$$f_{AR}(\lambda) = \frac{\sigma^2}{2\pi |A(e^{i\lambda})|^2} \tag{2.36a}$$

$$= \frac{\sigma^2}{2\pi \prod_{i=1}^{M} (1-p_i e^{-i\lambda})(1-p_i e^{i\lambda})} \tag{2.36b}$$

The spectral representation for the output covariance sequence (2.25) is

$$C_{AR}(k) = \int_{-\pi}^{\pi} e^{i\lambda k} \, f_{AR}(\lambda) d\lambda =$$

$$= \frac{\sigma^2}{2\pi j} \oint_{\Gamma} \frac{z^{k-1} z^M}{\prod_{i=1}^{M} (z-p_i)(1-p_i z)} dz \tag{2.37}$$

where Γ is a counter clockwise contour lying in the region of absolute convergence of the integral. With the residue theorem we then find for nonnegative lags, and the assumption of simple poles:

$$C_{AR}(k) = \sigma^2 \sum \text{residues inside } \Gamma$$

$$= \sum_{\ell=1}^{M} \lim_{z \to p_\ell} \frac{(z-p_\ell)\sigma^2 z^{k+M-1}}{\prod_{i=1}^{M} (z-p_i)(1-p_i z)} \tag{2.38}$$

$$= \sum_{\ell=1}^{M} \frac{\sigma^2 p_\ell^{M-1}}{(1-p_\ell^2) \prod_{\substack{i=1 \\ \neq \ell}}^{M} (p_\ell - p_i)(1-p_i p_\ell)} p_\ell^k \tag{2.39}$$

For negative lags use the definition $C_{AR}(-k) = C_{AR}(k)$.

Assume given a moving average system with real coefficients b_i and with transfer function

$$H_{MA}(z) = 1 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_N z^{-N} \tag{2.40a}$$

$$= B(z) \tag{2.40b}$$

$$= \prod_{j=1}^{N} (1 - q_j z^{-1}) \tag{2.40c}$$

When the above MA system is driven by a white noise sequence $(0,\sigma^2)$, the output spectral density is given by

$$f_{MA}(\lambda) = \frac{\sigma^2}{2\pi} \left| B(e^{i\lambda}) \right|^2 \tag{2.41a}$$

$$= \frac{\sigma^2}{2\pi} \sum_{i=0}^{N} b_i e^{-i\lambda i} \sum_{j=0}^{N} b_j e^{i\lambda j} \tag{2.41b}$$

Consequently the spectral representation for the corresponding covariance sequence is

$$C_{MA}(k) = \int_{-\pi}^{\pi} e^{i\lambda k} f_{MA}(\lambda) d \qquad 2.42a$$

$$= \frac{\sigma^2}{2\pi j} \oint_{\Gamma} z^{k-1} \sum_{i=0}^{N} b_i z^{-i} \sum_{j=0}^{N} b_j z^{j} \, dz \qquad 2.42b$$

$$= \frac{\sigma^2}{2\pi j} \oint_{\Gamma} z^{k-1-N} \sum_{i=0}^{N} b_i z^{N-i} \sum_{j=0}^{N} b_j z^{j} \, dz \qquad 2.42c$$

Using again the residue theorem this leads to the familiar pattern

$$C_{MA}(k) = \begin{cases} \sigma^2 (\text{res at } z=0) & 0 \le k \le N \\ 0 & k \ge N+1 \end{cases} \qquad 2.43$$

For lags $0 \le k \le N$ the pole at $z=0$ is of order $n = N+1-k$ and the corresponding $C_{MA}(k)$ is found according to

$$C_{MA}(k) = \sigma^2 \lim_{z \to 0} \frac{d^{n-1}\{ \sum_{i=0}^{N} b_i z^{N-i} \sum_{j=0}^{N} b_j z^{j} \}}{(n-1)! \, dz^{n-1}} \qquad 2.44a$$

$$= \sigma^2 \sum_{i=0}^{N} b_i \, b_{i+k} \qquad 2.44b$$

Again use $C_{MA}(-k) = C_{MA}(k)$ to define the covariance sequence for negative lags.

Assume given a stable autoregressive moving average ARMA(M,N) system with real numerator and denominator coefficients $\{b_j\}_0^N$ and $\{a_j\}_1^M$ respectively, and simple poles. The transfer function can be written:

$$H_{ARMA(M,N)}(z) = \frac{1+b_1 z^{-1}+\ldots+b_N z^{-N}}{1+a_1 z^{-1}+\ldots+a_M z^{-M}} \qquad 2.45a$$

$$= \frac{B(z)}{A(z)} \qquad 2.45b$$

$$= \frac{\prod_{j=1}^{N} (1-q_j z^{-1})}{\prod_{i=1}^{M} (1-p_i z^{-1})} \quad ; \quad |p_i| < 1 \qquad 2.45c$$

Driven by a white noise sequence $(0,\sigma^2)$ the resulting output spectral density is given by

$$f_{ARMA(M,N)}(\lambda) = \frac{\sigma^2}{2\pi} \left|\frac{B(e^{i\lambda})}{A(e^{i\lambda})}\right|^2 \qquad 2.46a$$

$$= \frac{\sigma^2}{2\pi} \frac{\prod_{j=1}^{N} (1-q_j e^{-i\lambda})(1-q_j e^{i\lambda})}{\prod_{i=1}^{M} (1-p_i e^{-i\lambda})(1-p_i e^{i\lambda})} \qquad 2.46b$$

The spectral representation for the corresponding covariance sequence yields

$$C_{ARMA(M,N)}(k) = \int_{-\pi}^{\pi} e^{i\lambda k} f_{ARMA(M,N)}(\lambda)\, d\lambda \qquad 2.47a$$

$$= \frac{\sigma^2}{2\pi j} \oint_{\Gamma} z^{k-1+M-N} \frac{\prod_{j=1}^{N} (z-q_j)(1-q_j z)}{\prod_{i=1}^{M} (z-p_i)(1-p_i z)}\, dz \qquad 2.47b$$

For $k \geq 0$ we derive via the residues for poles inside Γ:

$$C_{ARMA(M,N)}(k) = \sigma^2 \sum_{\ell=1}^{M} \lim_{z \to p_\ell} (z-p_\ell) \frac{z^{k-1+M-N} \prod_{j=1}^{N} (z-q_j)(1-q_j z)}{\prod_{i=1}^{M} (z-p_i)(1-p_i z)}$$

$$+ \sigma^2 \lim_{z \to 0} \frac{d^{n-1}\{z^n I(z)\}}{(n-1)!\, dz^{n-1}} I(n) \qquad 2.48$$

where $I(z)$ in the integrand in (2.47b) and $n=N-M+1-k$ relates to a pole at $z=0$ of order n. The indicator function $I(n)$ is zero except when $n>0$.

Therefore,

$$C_{ARMA(M,N)}(k) = \sum_{\ell=1}^{M} \frac{\sigma^2 p_\ell^{M-N-1} \prod_{j=1}^{N} (p_\ell - q_j)(1-q_j p_\ell)}{(1-p_\ell^2) \prod_{\substack{i=1 \\ i\neq\ell}}^{M} (p_\ell - p_i)(1-p_i p_\ell)} p_\ell^k$$

$$+ \lim_{z\to 0} \frac{\sigma^2 d^{n-1}}{(n-1)!\,dz^{n-1}} \left\{ \frac{\prod_{j=1}^{N} (z-q_j)(1-q_j z)}{\prod_{i=1}^{M} (z-p_i)(1-p_i z)} \right\} I(n) \qquad 2.49$$

The covariance sequence symmetry property again defines the covariance sequence for negative lags. From the above results we deduce that the covariance sequence for an ARMA(M,N) process with simple poles can be represented in the form of a complex linear combination of complex exponentials whenever the numerator order N is at most one less than the denominator order M, which we will denote M^- in the sequel. For these so-called strictly proper rational transfer functions the following covariance sequence model is valid:

$$C_{ARMA}(k) = \sum_{\ell=1}^{M} A_\ell \, p_\ell^{\,|k|} \quad ; \quad \forall\, k \qquad 2.50$$

Because the coefficients $\{a_i\}_1^M$ and $\{b_i\}_0^N$ are real, we have the property that $\{A_\ell, p_\ell\}_1^M$ are either real or occur in complex conjugate pairs.

At this point we note that in particular the covariance sequence expressions (2.39) and (2.49) were derived to show when expression (2.50) would be valid. If one desires to compute numerically the covariance sequence for a general ARMA(M,N) system driven by white noise, it is much more convenient to solve the linear set of generalized normal equations of order M, extend the resulting sequence to

order max(M,N) and subsequently use the autoregressive recursion of the denominator to compute as many of the positive going lags as desired. This approach is discussed in [DBE].

2.3 The DFT and its Synthesis Process Spectral Representation

Assume we have the following partial sequence realization x(0), x(1),...,x(N-1). For the periodic extension of this partial realization write the Fourier series expansion as follows

$$X(\lambda_n) = \sum_{k=0}^{N-1} x(k) e^{i\lambda_n k} \qquad 2.51a$$

where

$$x(k) = \frac{1}{N} \sum_{n=0}^{N-1} X(\lambda_n)\, e^{-i\lambda_n k} \qquad 2.51b$$

with

$$\lambda_n = \frac{2\pi n}{N} \quad ; \quad 0 \leq n \leq N-1$$

These relations constitute the Discrete Fourier Transform (DFT) pair, which can be implemented very efficiently on a digital machine, due to the symmetry properties of $e^{i\lambda_n k}$ and $e^{i\lambda_{N-n} k}$. The algorithms referred to are known as FFT algorithms. Recall that the spectral representation for a stochastic almost periodic function was given in (2.19):

$$V(t) = \sum_{j=-n}^{n} Z_j\, e^{i\lambda_j t} \quad ; \quad -\infty < t < \infty$$

Due to the symmetry property of the $\{\lambda_j\}$ and the orthogonality property of the $\{Z_j\}$, as in (2.20), we recognize that x(k) in (2.51b) is a particular realization of the stochastic almost periodic process with harmonically related frequencies λ_j. The power at frequency λ_j is reflected in $E\{|X(\lambda_j)/N|^2\}$. For a particular partial realization of

a process, the DFT therefore merely yields an estimate of the power at the harmonically related frequencies λ_j.

It is worthwhile to note that the covariance sequence corresponding to the analysis model underlying the DFT is represented by a positive real linear combination of harmonically related complex exponentials as in (2.22). DFT spectral analysis or the periodogram is therefore recognized as a special case of the model presented in (2.50).

2.4 Pisarenko Decomposition and its Synthesis Process Spectral Representation

Let us assume a very practical situation where we have a stochastic almost periodic process X(k) in additive noise N(k), that is orthogonal to the process X(k). The covariance sequence is now given by

$$\begin{aligned} C(k) &= E\{(X_{\ell+k} + N_{\ell+k})(X_\ell^* + N_\ell^*)\} \\ &= E\{X_{\ell+k}X_\ell^*\} + E\{X_{\ell+k}N_\ell^*\} + E\{N_{\ell+k}X_\ell^*\} + E\{N_{\ell+k}N_\ell^*\} \\ &= C_X(k) + C_N(k) \end{aligned} \qquad 2.52$$

$C_X(k)$ is given by (2.22) and $C_N(k)$ depends on the dynamic properties of the noise generating process. For the particular case where the assumption is made that N(k) represents a white noise process, the above results in the model underlying the Pisarenko decomposition [P].

$$C_p(k) = \sigma_N^2\,\delta(k) + \sum_{j=0}^{n} \rho_j \cos\lambda_j k \qquad 2.53$$

The Pisarenko decomposition is thus seen to decompose a finite covariance sequence realization into a white noise component and a positive real linear combination of complex exponentials which are not necessarily harmonically related. If the underlying covariance sequence generator, or the process, satisfies the assumed model, then the parameters in (2.53) can be determined uniquely according to a theorem by Carathéodory

[GS]. The stochastic process synthesis equation can obviously be written by inspection as follows:

$$X_p(k,\omega) = \sum_{j=-n}^{n} Z_j(\omega) e^{i\lambda_j k} + N(k,\omega) \qquad 2.54$$

with

$$E|Z_j|^2 = \begin{cases} \rho_0 & j=0 \\ \rho_j/2 & j\neq 0 \end{cases} \quad ; \quad E|N|^2 = \sigma_N^2$$

Referring again to the model in (2.50), we note that a real pole p_ℓ can approximate arbitrarily closely the white noise imposed $\delta(k)$ when the pole radius goes to zero. The model underlying the Pisarenko decomposition then becomes a special case of the model presented in (2.50) as well.

2.5 A Generalized Synthesis Equation for Wide Sense Stationary Processes

Now the interesting question arises as to whether the stochastic process synthesis structures as indicated for periodogram analysis and Pisarenko decomposition, can be extended so as to allow damped complex exponentials in the spectral representation. Let us therefore investigate the following stochastic process synthesis equations:

$$X(k,\omega) = \sum_{m=1}^{M} \nu_{m,k}(\omega) e^{j\phi_m(\omega)} e^{j\lambda_m k} \quad ; \quad \lambda_m = 2\pi f_m T \qquad 2.55a$$

Here the complex random variable $\nu_{m,k}(\omega)$ is orthogonal to the random phase variable $\phi_m(\omega)$, and satisfies a first order autoregression [S]:

$$\nu_{m,k}(\omega) = \beta_m \nu_{m,k-1}(\omega) + \mu_m \varepsilon_{m,k} \qquad 2.55b$$

We furthermore choose

$$E\{\varepsilon_{m,k} \varepsilon^*_{\ell,j}\} = \frac{1-|\beta_m|^2}{|\mu_m|} \delta(m-\ell)\delta(k-j) \qquad 2.55c$$

Let us determine the covariance properties for $\nu_{m,k}(\omega)$, under the assumption that $\nu_{m,0}(\omega)$ is orthogonal to $\varepsilon_{\ell,k}$ for all ℓ,k:

$$E\{\nu_{m,k+j}(\omega)\nu^*_{m,k}(\omega)\} =$$

$$= E\{(\beta_m^{k+j}\nu_{m,0}(\omega) + \sum_{n=0}^{k+j-1}\beta_m^n\mu_m\varepsilon_{m,k+j-n})(\beta_m^{*k}\nu^*_{m,0}(\omega) + \sum_{\ell=0}^{k-1}\beta_m^{*\ell}\mu_m^*\varepsilon_{m,k-\ell})\}$$

$$= \beta_m^{k+j}\beta_m^{*k}E\{\nu_{m,0}(\omega)\nu^*_{m,0}(\omega)\} + \sum_{\ell=0}^{k-1}\sum_{n=0}^{k+j-1}\beta_m^n\beta_m^{*\ell}|\mu_m|^2E\{\varepsilon_{m,k+j-n}\varepsilon^*_{m,k-\ell}\}$$

$$= \beta_m^j|\beta_m|^{2k}E\{\nu_{m,0}(\omega)\nu^*_{m,0}(\omega)\} + \sum_{\ell=0}^{k-1}\sum_{n=0}^{k+j-1}\beta_m^n\beta_m^{*\ell}|\mu_m|^2\sigma_\varepsilon^2\delta(n-j-\ell)$$

$$= \beta_m^j|\beta_m|^{2k}E\{\nu_{m,0}(\omega)\nu^*_{m,0}(\omega)\} + \sum_{\ell=0}^{k-1}\beta_m^j|\beta_m|^{2\ell}|\mu_m|(1-|\beta_m|^2)$$

$$= \beta_m^j|\beta_m|^{2k}E\{\nu_{m,0}(\omega)\nu^*_{m,0}(\omega)\} + \beta_m^j|\mu_m|(1-|\beta_m|^2)\frac{1-|\beta_m|^{2k}}{1-|\beta_m|^2}$$

$$= \beta_m^j|\mu_m| + \beta_m^j|\beta_m|^{2k}[E\{\nu_{m,0}(\omega)\nu^*_{m,0}(\omega)\} - |\mu_m|] \qquad 2.56$$

Covariance sequence stationarity for $\nu_{m,k}(\omega)$ is seen to be obtained with

$$E\{\nu_{m,0}(\omega)\nu^*_{m,0}(\omega)\} = |\mu_m| \qquad 2.57$$

From (2.56) and wide sense stationarity of $\nu_{m,k}(\omega)$ we note:

$$E\{\nu_{m,k+j}(\omega)\nu^*_{\ell,k}(\omega)\} = \beta_m^j|\mu_m|\delta(m-\ell) \qquad 2.58$$

The autoregression of (2.55b) furthermore yields:

$$E\{\nu_{m,\ell+k}(\omega)\nu^*_{m,\ell}(\omega)\} =$$

$$= E\{\beta_m^k\nu_{m,\ell}(\omega)\nu^*_{m,\ell}(\omega) + \sum_{j=0}^{k-1}\beta_m^j\mu_m\varepsilon_{m,k-j+\ell}\nu^*_{m,\ell}(\omega)\}$$

$$= \beta_m^k E\{\nu_{m,\ell}(\omega)\nu^*_{m,\ell}(\omega)\}$$

$$= \beta_m^k|\mu_m| \qquad 2.59$$

For the covariance sequence of process generator (2.55a) we finally derive:

$$C(k) = E\{X(\ell+k,\omega)X^*(\ell,\omega)\}$$

$$= E\{\sum_{m=1}^{M} \nu_{m,\ell+k}(\omega)e^{j\phi_m(\omega)}e^{j\lambda_m(\ell+k)} \sum_{j=1}^{M} \nu^*_{j,\ell}(\omega)e^{-j\phi_j(\omega)}e^{-j\lambda_j\ell}\}$$

$$= \sum_{m=1}^{M} E\{\nu_{m,\ell+k}(\omega)\nu^*_{m,\ell}(\omega)\}e^{j\lambda_m k}$$

$$= \sum_{m=1}^{M} |\mu_m|\beta_m^k e^{j\lambda_m k}$$

$$= \sum_{m=1}^{M} |\mu_m||\beta_m|^k e^{j(\lambda_m+\arg\beta_m)k} \qquad 2.60$$

With $\{\nu_{m,k}(\omega)e^{j\phi_m(\omega)}e^{j\lambda_m k}\}$ real or in complex conjugate pairs the process $X(k,\omega)$ is real. Note from (2.60) that the linear weight involved is positive real, and with the generator equations (2.55) we can therefore synthesize a weakly stationary stochastic process in the class of ARMA(M,N) processes, which exhibits covariance sequences that are positive real linear combinations of damped complex exponentials. This point may be clarified as follows. Consider an ARMA(M,M) system with real coefficients and simple poles, so that its impulse response may be written:

$$h_k = \sum_{j=1}^{M} B_j p_j^k \qquad \forall\, k \geq 0 \qquad 2.61$$

For the covariance function of the stable system we derive:

$$C_k = \sum_{\ell=0}^{\infty} h_\ell \, h_{\ell+|k|}$$

$$= \sum_{i=1}^{M} B_i \sum_{j=1}^{M} B_j \sum_{\ell=0}^{\infty} p_i^{\ell} \, p_j^{\ell+|k|}$$

$$= \sum_{j=1}^{M} [B_j \sum_{i=1}^{M} B_i \frac{1}{1-p_i p_j}] \, p_j^{|k|} \qquad 2.62a$$

$$= \sum_{j=1}^{M} [\frac{|B_j|^2}{1-|p_j|^2} + B_j \sum_{\substack{i=1 \\ \neq *}}^{M} \frac{B_i}{1-p_i p_j}] \, p_j^{|k|} \qquad 2.62b$$

where $\neq *$ denotes the exclusion of i such that $B_i = B_j^*$. Note that by defining the term in square brackets of (2.62a) to equal A_j the general covariance sequence model of (2.50) for strictly proper ARMA(M,M^-) systems has been derived in an alternative way. The important interpretation of (2.62b) is that the second term in square brackets represents interactions between the different modes of the ARMA(M,M^-) system, resulting in a linear weight coefficient that is not necessarily positive real. Thus these interactions may not be represented in the synthesis model of (2.60). For a real process $\{\beta_m e^{j\lambda_m}\}$ are either real or occur in complex conjugate pairs, and (2.60) thereby represents a positive real linear combination of nonnegative definite covariance functions [Y], which implies that C(k) of (2.60) is nonnegative definite [W]. By the Herglotz theorem it is only for nonnegative definite functions that the spectral representation of (2.10) exists [BU].

From (2.62b) we see that (2.60) does not represent the most general linear combination of damped exponentials that represents a nonnegative definite covariance function. A further generalization would be achieved when $|\mu_m|$ in (2.60) were replaced by real or complex conjugate pairs of

linear weights such as indicated in (2.50) derived from strictly proper ARMA(M,N) systems driven by white noise. The gain derived from the more general representation of (2.50) is illustrated by the spectral representation theorem for wide sense stationary processes as given by Yaglom [Y]:

> Given any wide sense stationary process $\xi(t)$, any $\varepsilon > 0$ and any $T > 0$, there exist random variables $\xi_1, \xi_2, \ldots, \xi_n$ which are uncorrelated in pairs (the number n depends on ε and T, of course) and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that
>
> $$E\left|\xi(t) - \sum_{k=1}^{n} \xi_k e^{i\lambda_k t}\right|^2 < \varepsilon \qquad 2.63$$
>
> for any $t \in [-T,T]$.

Every wide sense stationary stochastic process can be approximated arbitrarily closely in mean square sense, by a stochastic almost periodic function, but to achieve better accuracy, in general the number of harmonic components has to be increased and the differences $\lambda_{k+1} - \lambda_k$ between neighboring frequencies has to be decreased.

For covariance sequences as derived above for strictly proper ARMA(M,N) systems increased accuracy would indeed force $n \to \infty$ and $\lambda_{k+1} - \lambda_k \to 0$ in (2.63). This would leave us with an infinite parameter representation for a clearly finite parameter problem indicated by (2.50). It is exactly in this finite parameter spectral representation that we find a powerful tool for parametric spectrum analysis (Chapter 4) and recursive digital filter design (Chapter 5).

2.6 Diagram for Generalized Stochastic Process Synthesis

Generation of the process $X(k,\omega)$ of (2.55) may be achieved by the implementation in Figure 2.1. The random variables can be considered real and are orthogonal:

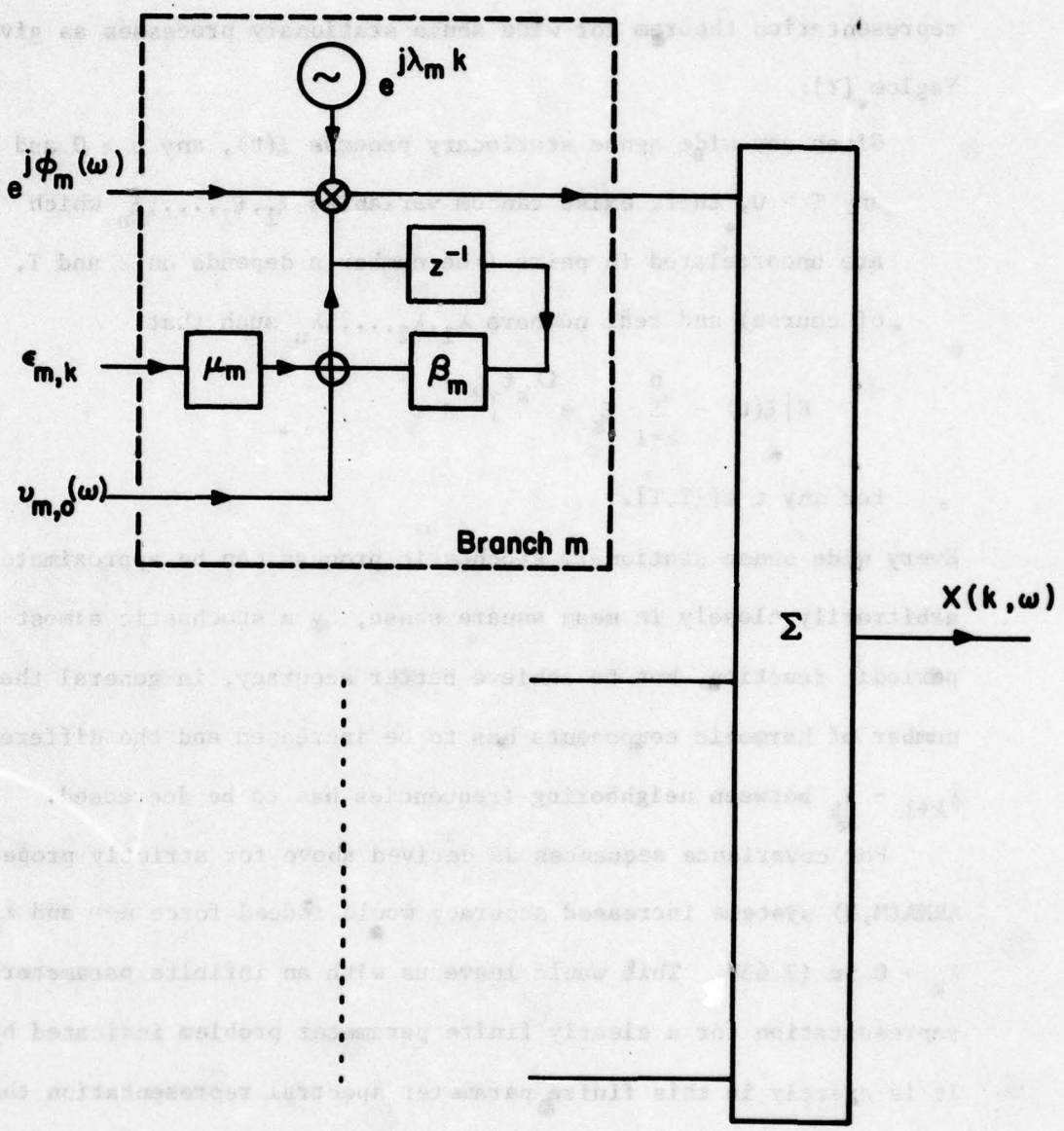

Figure 2.1 Generalized stochastic process synthesis.

$$\phi_m(\omega) \perp\!\!\!\perp \varepsilon_{m,k}(\omega) \perp\!\!\!\perp \nu_{m,0}(\omega) \qquad 2.64$$

Their respective statistics are given by:

$$\phi_m(\omega) \sim U[0,2\pi] \qquad 2.65a$$

$$\varepsilon_{m,k}(\omega) \sim (0, \frac{1-|\beta_m|^2}{|\mu_m|}) \qquad 2.65b$$

$$\nu_{m,0}(\omega) \sim (0, |\mu_m|) \qquad 2.65c$$

The independent branches each generate a real process, or they occur in independent complex conjugate pairs so that the pair generates a real process. The sum of all the branches is then real. The parameter choices for the particular process synthesis equations are as follows:

For DFT related process synthesis:

$$\text{all } \beta_m = 1$$

$$\lambda_m = \frac{2\pi m}{N} \quad ; \quad 0 \leq m \leq N-1 \qquad 2.66$$

For Pisarenko decomposition related process synthesis:

$$\beta_m = 1 \quad \forall\, m \neq 0$$

λ_m not nessarily harmonically related

white noise branch for m = 0:

$$\lambda_0 = 0$$

$$\phi_0(\omega) = 0$$

$$\beta_0 = 0 \text{ (pole at z=0)}$$

$$\nu_{0,0}(\omega) = 0$$

$$\mu_0 \text{ real} \qquad 2.67$$

The generalized process synthesis is indicated in Figure 2.1. Note that if $\{\lambda_m\}$ are to indicate the constituent frequencies, $\{\beta_m\}$ should be taken real in (2.60), which obviously does not form a restriction.

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3 A MODIFIED LEAST SQUARES PROBLEM AND ITS APPLICATION TO COVARIANCE SEQUENCE APPROXIMATION

First we pose the general least squares problem for approximating covariance sequences and discuss several ways to solve or avoid it.

In Section 3.2 the approach taken in this thesis is outlined. First the complex exponentials for the proposed covariance sequence model are found by minimizing the covariance sequence linear prediction errors. Subsequently the corresponding linear combination weights are determined in a general least squares fashion. This necessarily sub-optimal approach leads to linear systems of equations and a polynomial rootfinding problem to be solved. The somewhat ad hoc nature of this approach precludes the derivation of a simple spectral error criterion.

A recursive solution to the covariance sequence predictor is derived in Section 3.3 and its desirable properties are enumerated in the final section.

3.1 General and Modified Least Squares Problems

For our covariance sequence model we choose the complex linear combination of damped complex exponentials found in the previous chapter:

$$R_k = \sum_{i=1}^{M} A_i p_i^{|k|} \quad \forall\, k \qquad 3.1$$

Clearly, $R_{-k} = R_k$, and as derived before $\{A_i, p_i\}$ are either real or occur in complex conjugate pairs. There are several reasons one might choose this model:

(1) Such a covariance sequence model arises naturally from a strictly proper ARMA(M,N) process driven by a white sequence.

(2) This model preserves second order statistical properties in converting from continuous time to discrete time. A strictly proper continuous time process described by an M^{th} order differential equation, transforms under those preservation conditions to a discrete time system with M poles and at most M-1 zeros [PS].

(3) For physically realizable systems it seems natural to represent the covariance sequence by a complex linear combination of complex exponentials.

(4) The model corresponds to a generalized stochastic process synthesis structure based on positive real linear combinations of damped complex exponentials.

In fitting the model of (3.1) in a least squares sense to a given symmetric sequence $\{C_k\}$, k=0, ±1, ±2,...,±L, a simultaneous minimization with respect to A_i and p_i has to be performed on the following error criterion:

$$E = \sum_{k=-L}^{L} \{C_k - R_k\}^2$$

$$= \sum_{k=-L}^{L} \{C_k - \sum_{i=1}^{M} A_i p_i^{|k|}\}^2 \qquad 3.2$$

Equating the partial derivatives of E to zero yields:

$$0 = \frac{\partial E}{\partial A_j} = \sum_{k=-L}^{L} \{C_k - \sum_{i=1}^{M} A_i p_i^{|k|}\} p_j^{|k|} \quad ; \quad j=1,2,\ldots,M \qquad 3.3a$$

and

$$0 = \frac{\partial E}{\partial p_j} = \sum_{k=-L}^{L} \{C_k - \sum_{i=1}^{M} A_i p_i^{|k|}\} A_j |k| p_j^{|k|-1} \quad ; \quad j=1,2,\ldots,M \qquad 3.3b$$

The above equations are the discrete time versions of the Aigrain-Williams equations [AW], with the distinction that they are derived here for approximating the two-sided covariance sequence $\{C_k\}$ instead of a one-sided causal impulse response sequence $\{h_k\}$. Note that equations (3.3a) are linear in $\{A_i\}_1^M$. For any given set of poles $\{p_i\}_1^M$ the corresponding linear combination weights $\{A_i\}_1^M$ can be determined in straightforward linear fashion by solving (3.3a):

$$\begin{bmatrix} p_1^L & p_1^{L-1} & \cdots 1 & p_1 \cdots & p_1^L \\ \vdots & & & & \vdots \\ p_M^L & p_M^{L-1} & \cdots 1 & p_M \cdots & p_M^L \end{bmatrix} \begin{bmatrix} C_{-L} \\ \vdots \\ C_0 \\ \vdots \\ C_L \end{bmatrix} = \begin{bmatrix} p_1^L & \cdots 1 & p_1 & \cdots p_1^L \\ \vdots & & & \vdots \\ p_M^L & \cdots 1 & p_M & \cdots p_M^L \end{bmatrix} \begin{bmatrix} p_1^L & \cdots & p_M^L \\ \vdots & & \vdots \\ 1 & & 1 \\ p_1 & & p_M \\ \vdots & & \vdots \\ p_1^L & & p_M^L \end{bmatrix} \begin{bmatrix} A_1 \\ \vdots \\ A_M \end{bmatrix} \tag{3.4}$$

Due to symmetry properties this system of equations can be reduced considerably:

$$\begin{bmatrix} 1 & 2p_1 & \cdots 2p_1^L \\ \vdots & & \vdots \\ 1 & 2p_M & \cdots 2p_M^L \end{bmatrix} \begin{bmatrix} C_0 \\ \vdots \\ C_L \end{bmatrix} = \begin{bmatrix} 1 & p_1 & \cdots p_1^L \\ \vdots & & \vdots \\ 1 & p_M & \cdots p_M^L \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ 2p_1 & & 2p_M \\ \vdots & & \vdots \\ 2p_1^L & \cdots & 2p_M^L \end{bmatrix} \begin{bmatrix} A_1 \\ \vdots \\ A_M \end{bmatrix} \tag{3.5a}$$

Define

$$\underline{C}^T = [C_0, \; C_1, \; \cdots \; , C_L]$$

$$\Lambda^T = [\Lambda_1, \; \Lambda_2, \; \cdots \Lambda_M]$$

$$P^T = \begin{bmatrix} 1 & p_1 & p_1^2 & \cdots & p_1^L \\ \vdots & & & & \vdots \\ 1 & p_M & p_M^2 & \cdots & p_M^L \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & & \underline{0} \\ & 2I & \\ \underline{0} & & \end{bmatrix} \qquad 3.5b$$

and write (3.3a) as:

$$P^T Q \, \underline{C} = P^T Q \, P \, \underline{A} \qquad 3.6$$

Then the solution for the linear weights is

$$\underline{A} = [P^T Q P]^{-1} P^T \, Q \, \underline{C} \qquad 3.7$$

which is recognized as a weighted least squares fit for covariance lags $k \geq 0$. The weighting is entirely due to the two-sided approximation of a real symmetric sequence with an approximating, real symmetric sequence. The approximation for C_0 turns up once, whereas the approximation for C_k, $k > 0$, turns up once again in the approximation of C_{-k}, resulting obviously in the weighting matrix Q above. The matrix P is a Vandermonde matrix and with the assumption that all roots $\{p_i\}_1^M$ are distinct P is full rank [C]. With the given definition for Q the determinant of $P^T Q P$ differs from zero and a unique solution for $\underline{A}$ results from (3.7) [ND].

The equations (3.3b) can be written in a similarly reduced form:

$$\acute{P}^T \, Q \, \underline{C} = \acute{P}^T \, Q \, P \, \underline{A} \qquad 3.8$$

Here the definitions for Q, $\underline{C}$, P and $\underline{A}$ remain as before, and $\acute{P}$ is defined as

$$\acute{P} = \sum_{j=1}^{M} A_j \frac{\partial P}{\partial p_j} \qquad 3.9$$

Thus

$$\sum_{j=1}^{M} A_j \frac{\partial P^T}{\partial p_j} Q \underline{C} = \sum_{j=1}^{M} A_j \frac{\partial P^T}{\partial p_j} Q P \underline{A} \qquad 3.10$$

This expression is nonlinear in $\{p_j\}$. In the very special case where the sequence $\{C_k\}$ has an exact representation of the form (3.1) the solution obviously satisfies:

$$\underline{C} = P \underline{A} \qquad 3.11$$

In the following section we will show how this solution can be obtained as a special case when the model exactly matches the desired covariance sequence $\{C_k\}$ over the entire interval of observation. The latter is the case when C_k satisfies an ARMA(M,M$^-$) model, or when the chosen model order M is so large that an exact match of C_k is obtained over the interval of approximation.

The nonlinear minimization problem outlined above generally requires iterative solution methods, several of which have been suggested. Based on geometric orthogonality concepts, an orthonormal basis can be generated iteratively, and the projection theorem would lead to the best approximation [MH]. Another iterative procedure arises from a system identification point of view [SMC], where an estimate $(A_{i-1}(z))$ for the denominator polynomial filter is used to prefilter in- and output of the system to be identified. The next estimates for denominator $(A_i(z))$ and numerator $(Q_i(z))$ polynomials are obtained simultaneously using either of the following equivalent generalized least squares formulations:

$$\min_{A_i, Q_i} \oint \left| \frac{Q_i(z)}{A_{i-1}(z)} - \frac{A_i(z)}{A_{i-1}(z)} H(z) \right|^2 \frac{dz}{z} \qquad 3.12a$$

$$\min_{A_i, Q_i} \oint \left| \frac{Q_i(z)}{A_i(z)} - H(z) \right|^2 \left| \frac{A_i(z)}{A_{i-1}(z)} \right|^2 \frac{dz}{z} \qquad 3.12b$$

The new denominator polynomial filter is used to iterate the procedure. Experimental results have shown that this process often converges quickly but also that the danger of divergence is imminent when too many parameters are specified [ST]. The latter may explain why a general convergence proof has not been given since appearance of the original paper. A similar linear iterative process with improved convergence properties has been suggested [M1], and requires only one mode of iteration. The relations between [MH] and [SMC] are described in [M2].

The criterion in (3.12) is based on a suggestion by Kalman [K] and is known loosely as the modified least squares criterion. A recent approach capitalizes on the resulting tractable minimization problem to find an approximating ARMA(M,N) system based on finite impulse response and covariance sequences [MR]. In the modified problem one seeks to minimize the following error with respect to denominator coefficients $\{a_i\}$ and numerator coefficients $\{q_j\}$:

$$E = \int_{-\pi}^{\pi} |H(e^{j\omega})A(e^{j\omega})-Q(e^{j\omega})|^2 \, d\omega \qquad 3.13$$

Note that the important difference with the familiar least squares formulation is the introduction of a spectral weighting function $|A(e^{j\omega})|^2$, which happens to be the magnitude squared response of the yet unknown denominator polynomial. The resulting error criterion represents a quadratic minimization problem. It leads to a condition for the numerator coefficients depending on impulse response data and the denominator coefficients. When this condition on the numerator coefficients is substituted in the error expression, it results in a constrained quadratic minimization problem for the denominator coefficients only. The authors give an efficient algorithm of Levinson type

to find the filter coefficients. The most important requirement for the above procedure to yield a stable solution, is that the finite sequences $\{h_k\}$ and $\{C_k\}$ be consistent. One way to take care of this requirement [SL] is to find a very high order autoregressive approximation first, using say the autocorrelation method of linear prediction [MA]. The resulting high order system is used to generate consistent impulse and covariance sequences, before applying the Mullis-Roberts algorithm to yield stable, low order ARMA(M,N) systems.

Time domain approaches to pole-zero modeling have been around for a long time. Prony devised a method for functional interpolation [PR], which was extended later to include least squares fits [H]. The original method by Prony was shown to be equivalent to Padé approximation [WM]. Later time domain approaches have been termed rediscoveries of Prony's method [SH], [BP]. These approaches are all based on the fact that the tail of the impulse response of an ARMA(M,N) system satisfies a pure autoregression. The latter property allows one to find the system poles first, and thereafter the zeros for given poles. These procedures have been very popular in spite of the fact that in general stability cannot be guaranteed for the resulting filters. The procedure in the following section is quite similar to the above time domain least squares approaches. Differences lie in the application to two-sided covariance sequence approximation and the possibility to carefully monitor the stability of the recursively developing denominator polynomial, so that stability can be guaranteed.

3.2 A Covariance Sequence Prediction Approach

The question arises as to whether the general nonlinear least squares problem can be approximated as far as the error criterion goes,

but avoided as far as iterative solution methods are concerned. Remember that the covariance sequence model arose naturally from an ARMA(M,M⁻) system driven by a white sequence. We can therefore write the difference equation for such a system:

$$\sum_{i=0}^{M} a_i y_{k-i} = \sum_{j=0}^{M-1} b_j x_{k-j} \quad ; \quad a_0 \triangleq 1 \qquad 3.14$$

The impulse response then satisfies the same difference equation

$$\sum_{i=0}^{M} a_i h_{k-i} = \sum_{j=0}^{M-1} b_j \delta_{k-j} \qquad 3.15$$

and with the definition

$$C_k = \sum_{i=0}^{\infty} h_i h_{i+k} \qquad 3.16$$

the covariance sequence for an ARMA(M,M⁻) filter is found [KPSS] to satisfy:

$$\sum_{i=0}^{M} a_i C_{k-i} = \sum_{j=0}^{M-1-k} h_j b_{j+k} \quad , \quad k \geq 0 \qquad 3.17$$

It is important to note that the covariance sequence of this ARMA(M,M⁻) system satisfies a purely autoregressive difference equation for lags $k \geq M$. We will use this property in order to find the denominator polynomial coefficients $\{a_i\}_0^M$, from which the poles $\{p_i\}_1^M$ can be found by polynomial rootfinding. From (3.5) note that if the complex exponentials are distinct, the $\{A_i\}_1^M$ can be found such that $C_0,\ldots,C_{M-1}$ are exactly matched. Under this condition the minimization problem of (3.2) gets replaced by

$$\min_{\{p_i\}} \sum_{k=M}^{L} \{C_k - R_k\}^2 \qquad 3.18$$

A weighting of these approximation errors by the autoregression of the polynomial corresponding to $\{p_i\}$ analogous to (3.13) leads to

$$\min_{\{a_i\}} \sum_{k=M}^{L} \{ \sum_{i=0}^{M} a_i C_{k-i} - \sum_{i=0}^{M} a_i R_{k-i} \}^2 \qquad 3.19$$

Recognize that the approximating sequence $\{R_k\}$ exactly satisfies the autoregressive recursion as in (3.17). An alternative interpretation of this procedure is therefore that we are looking for the best linear predictor for the covariance sequence:

$$\min_{\{a_i\}} \sum_{k=M}^{L} \{C_k + \sum_{i=1}^{M} a_i C_{k-i}\}^2 \qquad 3.20$$

Taking derivatives with respect to $\{a_j\}$ leads to the following system of linear equations:

$$\sum_{k=M}^{L} \{C_k + \sum_{i=1}^{M} a_i C_{k-i}\} C_{k-j} = 0 \quad ; \quad j=1,2,\ldots,M \qquad 3.21$$

Let us write them out:

$$\begin{bmatrix} C_{M-1} & C_M \cdots C_{L-1} \\ C_{M-2} & \vdots \\ \vdots & \\ C_0 & C_1 \cdots C_{L-M} \end{bmatrix} \begin{bmatrix} C_{M-1} & C_{M-2} \cdots C_0 \\ C_M & C_{M-1} \quad C_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ C_{L-1} & C_{L-2} \cdots C_{L-M} \end{bmatrix} \begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_M \end{bmatrix} = - \begin{bmatrix} C_{M-1} \cdots C_{L-1} \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ C_0 \cdots C_{L-M} \end{bmatrix} \begin{bmatrix} C_M \\ C_{M+1} \\ \cdot \\ \cdot \\ \cdot \\ C_L \end{bmatrix} \qquad 3.22a$$

Define:

$$C = \begin{bmatrix} C_{M-1} & C_{M-2} \cdots C_0 \\ C_M & \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ C_{L-1} & C_{L-2} \cdots C_{L-M} \end{bmatrix}$$

$$\underline{a}^T = [a_1, \ a_2, \ \ldots \ , a_M]$$

$$\underline{c}^T = [C_M, \ C_{M+1}, \ \ldots \ , C_L] \qquad 3.22b$$

which yields the following matrix equation

$$C^T C \ \underline{a} = -C^T \underline{c} \qquad 3.23$$

The elements of $C^T C$ are covariance estimates of the covariance sequence, given by

$$\{C^T C\}_{ij} = \sum_{k=M}^{L} C_{k-i} \ C_{k-j} \ ; \ \ 1 \leq i,j \leq M \qquad 3.24$$

and it is easily proved that $C^T C$ is nonnegative definite. Of course (3.23) can be solved by general linear systems routines, but its symmetry can be used to advantage and the more efficient Choleski decomposition performed. The linear system of equations (3.23) has a unique solution when the determinant of $C^T C$ does not equal zero, a condition met if and only if C is full rank. In case the underlying system for C_k has order less than M, the leftmost column of C can be expressed as a linear combination of the other columns and C is not full rank. Many solutions would be possible in that case. We note here that in practical cases $\{C_k\}$ does not exactly satisfy an Mth order difference equation and (3.23) will have a unique least squares solution. Furthermore, even when $C^T C$ is singular, due to an over estimation of the underlying autoregressive system order, the event can be effectively anticipated by a scheme that recursively updates the autoregressive coefficients and the system order. By keeping track of the covariance sequence prediction error at each step, singularity of $C^T C$ can be detected. We point out here that (3.23) represents the covariance method of linear prediction for predicting the tail ($k \geq M$) of an otherwise symmetric real covariance sequence.

The covariance method of linear prediction can be derived under the assumption that the portion of the signal from which the predictor parameters are computed is nonstationary [MW]. Let $\{R_c(\ell,k)\}$ denote the nonstationary covariance function for the covariance sequence $\{C_k\}_0^L$. Let $\{R_e(\ell,k)\}$ denote the covariance function for the error sequence $\{e_k\}$ observed when the linear prediction filter A(z) is excited by $\{C_k\}$. Thus $R_c(\ell,k)$ may be written

$$R_c(\ell,k) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} Q_c(\omega,\omega') e^{j(\omega\ell-\omega' k)} d\omega d\omega' \qquad 3.25$$

with

$$Q_c(\omega,\omega') = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} R_c(\ell,k)\, e^{-j(\omega\ell-\omega' k)} \qquad 3.26$$

It follows easily [PA] that

$$Q_e(\omega,\omega') = Q_c(\omega,\omega') A(e^{j\omega}) A^*(e^{j\omega'}) \qquad 3.27$$

Thus the error variance $E = R_e(0,0)$ may be written

$$E = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} Q_c(\omega,\omega') A(e^{j\omega}) A^*(e^{j\omega'}) d\omega d\omega' \qquad 3.28$$

With substitution of the linear prediction filter

$$E = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} Q_c(\omega,\omega') [1 + \sum_{k=1}^{M} a_k e^{-j\omega k}][1 + \sum_{\ell=1}^{M} a_\ell e^{j\omega'\ell}] d\omega d\omega' \qquad 3.29$$

Minimizing E with respect to the predictor coefficients yields:

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} Q_c(\omega,\omega') [e^{-ji\omega}\{1+\sum_{\ell=1}^{M} a_\ell e^{j\ell\omega'}\} + e^{ji\omega'}\{1+\sum_{k=1}^{M} a_k e^{-j\omega k}\}] d\omega d\omega' = 0; \quad i=1,2,\ldots,M \qquad 3.30$$

The definition of (3.26) allows us to write

$$\sum_{k=1}^{M} a_k R_c(-i,-k) = -R_c(-i,0) \; ; \; 1 \leq i \leq M \qquad 3.31$$

Equations (3.31) are known as the generalized normal equations. With $\hat{Q}_c(\omega,\omega')$ chosen to be

$$\hat{Q}_c(\omega,\omega') = \frac{\sigma^2}{A(e^{j\omega})A^*(e^{j\omega'})} \qquad 3.32$$

the spectral error criterion can be rewritten as

$$E = \left(\frac{\sigma}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{Q_c(\omega,\omega')}{\hat{Q}_c(\omega,\omega')} \, d\omega d\omega' \qquad 3.33$$

Minimization with respect to the predictor coefficients of either (3.29) or (3.33) is seen to yield the solution of (3.31). Similarities are present here with the analysis-by-synthesis procedures when approximating with a 2D all-pole model. Note that if $Q_c(\cdot,\cdot)$ is larger than $\hat{Q}_c(\cdot,\cdot)$ a significant error contribution results. If on the other hand, $Q_c(\cdot,\cdot)$ is smaller than $\hat{Q}_c(\cdot,\cdot)$ a relatively small error contribution takes place. As a result, the best approximation will occur where $Q_c(\cdot,\cdot)$ is high. Another observation to be made relates to the uniformity of the error criterion. Due to the ratio of $Q_c(\cdot,\cdot)$ over $\hat{Q}_c(\cdot,\cdot)$, matching should be uniform over the frequency domain of interest.

In approximating the nonstationary covariance of covariances $R_c(-i,-k)$ by $\{C^TC\}_{i,k}$ and substituting in (3.31), the equations (3.23) are derived. Thus the covariance method of linear prediction can be derived from a frequency domain formulation where the 2D spectrum of a nonstationary signal is to be approximated by an all-pole 2D spectrum. Under the assumption of a stationary signal the generalized formulation reduces to the autocorrelation method of linear prediction. Note that

an interpretation of the above spectral error criterion is not obvious, due to the fact that $Q_c(\cdot,\cdot)$ and $\hat{Q}_c(\cdot,\cdot)$ are spectra for the positive going tail of a covariance sequence used as a data sequence, so to speak. The element of C^TC in (3.24) can therefore be interpreted as covariance estimates for a covariance sequence. When one applies the covariance method of linear prediction to a data sequence of increasing length, its solution approaches the solution for the autocorrelation method of linear prediction. The latter analogy for covariance sequence prediction cannot hold because the covariance sequence estimate for an ARMA(M,M⁻) process should be considered representing a nonstationary signal. One should therefore be cautious in extrapolating known linear prediction results to the covariance sequence linear prediction practiced here. We will, however, transplant a nice recursive solution procedure for (3.23) to the covariance sequence environment, in the following section.

Assume that the solution to (3.23) is found so that the poles of the ARMA(M,M⁻) system can be found by polynomial rootfinding:

$$\sum_{i=0}^{M} a_i z^{-i} = \prod_{j=1}^{M} (1-p_j z^{-1}) \quad ; \quad a_0 \triangleq 1 \qquad 3.34$$

One way to solve for all the roots of a polynomial is by the deflation procedure as used in Muller's algorithm [CB]. An interpretation is that the nonlinear system of equations (3.3) for the general least squares covariance sequence approximation problem is now concentrated in the single algebraic equation of (3.34), which is a considerable reduction in complexity indeed.

The covariance sequence approximation procedure proposed here therefore consists of the following steps:

i) Find autoregressive prediction parameters by solving the linear system of equations (3.23)

ii) Compute the corresponding poles according to the nonlinear algebraic equation in (3.34)

iii) Find corresponding linear weights in a least squares fashion by solving the linear system of equations in (3.7)

All but the least squares fitting aspects of this procedure have been known for some two hundred years in the function interpolation context as Prony's method [PR]. Note that the last step is a departure from matching conditions used in the derivation of the covariance sequence prediction approach. Note also, however, the compliance with the general least squares approach, assuming the poles are known. Qualitatively this should move the procedure from a modified least squares approach in the direction of the least squares approach. A special case arises when C_k satisfies an ARMA(M,M$^-$) system of equations for then its representation is found exactly. One such case is in preserving second order statistical properties from a continuous time system in a discrete time implementation. The present method serves therefore as another alternative procedure in the design of covariance invariant digital filters [PS], [KPSS].

Note that the system of equations in (3.7) is written in complex form. Recognizing that the elements of the matrix P and vector $\underline{c}$ are real or occur in complex conjugate pairs enables us to rewrite the linear system of complex equations (3.7) as a linear system of real equations. Complex conjugate columns in P corresponding to $p_i=\rho e^{j\beta}$ and $p_{i+1}=\rho e^{-j\beta}$ get thereby replaced as follows:

$$P\,\underline{A} = \begin{bmatrix} \cdot & 1 & 1 & \cdot \\ \cdot & & & \cdot \\ \cdot & p_i & p_i^* & \cdot \\ \cdot & & & \cdot \\ \cdot & p_i^2 & p_i^{*2} & \cdot \\ \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & p_i^L & p_i^{*L} & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ A_i \\ A_i^* \\ \cdot \\ \cdot \end{bmatrix} \sim \begin{bmatrix} \cdot & 1 & 0 & \cdot \\ \cdot & & & \cdot \\ \cdot & \rho\cos\beta & \rho\sin\beta & \cdot \\ \cdot & & & \cdot \\ \cdot & \rho^2\cos 2\beta & \rho^2\sin 2\beta & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & \rho^L\cos L\beta & \rho^L\sin L\beta & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ B_i \\ B_{i+1} \\ \cdot \\ \cdot \end{bmatrix} = P_R\underline{B} \qquad 3.35$$

The elements of P_R and $\underline{B}$ are now real. We furthermore can relate the elements of $\underline{A}$ to the elements of $\underline{B}$:

$$A_i = \frac{B_i - jB_{i+1}}{2} \quad ; \quad A_{i+1} = A_i^* \qquad 3.36$$

For real p_j leave column in P and element A_j unchanged.

3.3 An Efficient Recursive Procedure for the Covariance Sequence Predictor

We will now take a closer look at a recently developed inner product formulation for minimizing prediction errors [MG]. This procedure will ultimately lead to fast computational algorithms and a sequential test for stability. The present development is based on [IS], where methods for implementing the autocorrelation method of linear prediction were presented. The order m of the polynomial will develop recursively, so m=1,2,...,M, where M is the preset maximum order for the predictor polynomial filter. Let us define the prediction error of interest, a forward prediction error:

$$c_m^+(k) = C_k - [-\sum_{i=1}^{m} a_{mi}\, C_{k-i}]$$

$$= \sum_{i=0}^{m} a_{mi}\, C_{k-i} \quad ; \quad a_{m0} \triangleq 1 \qquad 3.37$$

In the development we also need a backward prediction error:

$$c_m^-(k) = C_{k-m-1} - [-\sum_{i=1}^{m} b_{mi}\, C_{k-i}]$$

$$= \sum_{i=1}^{m+1} b_{mi}\, C_{k-i} \quad ; \quad b_{m,m+1} \triangleq 1 \qquad 3.38$$

The total forward and backward prediction errors can now be represented as

$$\alpha_m = \sum_{k=M}^{L} [c_m^+(k)]^2 \qquad 3.39a$$

$$\beta_m = \sum_{k=M}^{L} [c_m^-(k)]^2 \qquad 3.39b$$

We see that $c_m^+(k)$ and $c_m^-(k)$ can be considered the outputs of two filters $A_m(z)$ and $B_m(z)$ having as common input sequence $\{C_k\}$, where

$$A_m(z) = \sum_{i=0}^{m} a_{mi}\, z^{-i} \quad ; \quad a_{m0} \triangleq 1 \qquad 3.40a$$

and

$$B_m(z) = \sum_{i=1}^{m+1} b_{mi}\, z^{-i} \quad ; \quad b_{m,m+1} \triangleq 1 \qquad 3.40b$$

Let us now define the inner product as the summation of the products of the output sequences $c_m^+(k)$ and $c_m^-(k)$, as illustrated in Figure 3.1. This inner product definition can be written as

$$\langle A_m(z), B_m(z) \rangle_{C^TC} = \sum_{k=M}^{L} \sum_{i=0}^{m} a_{mi}\, C_{k-i} \sum_{j=1}^{m+1} b_{mj}\, C_{k-j} \qquad 3.41a$$

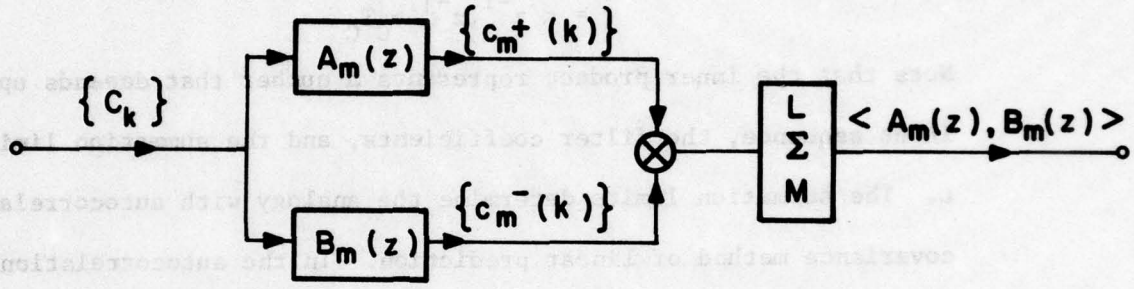

Figure 3.1 Filtering interpretation of inner product.

$$= \sum_{i=0}^{m} \sum_{j=1}^{m+1} a_{mi} \sum_{k=M}^{L} C_{k-i}\, C_{k-j}\, b_{mj} \qquad 3.41b$$

We readily recognize the terms of C^TC as defined in (3.24), which can be interpreted as covariance of covariance estimates:

$$\sum_{k=M}^{L} C_{k-i}\, C_{k-j} = \{C^TC\}_{ij}$$

$$= \langle z^{-i}, z^{-j} \rangle_{C^TC} \qquad 3.42$$

Note that the inner product represents a number that depends upon the input sequence, the filter coefficients, and the summation limits M and L. The summation limits determine the analogy with autocorrelation and covariance method of linear prediction. In the autocorrelation method we extend the summation limits indefinitely, which makes it possible to interpret the inner product in terms of integrals in the z-plane or the frequency domain. Only in the autocorrelation method is the problem of minimizing prediction errors isomorphic with the problem of polynomial approximation on the unit circle [KA]. Using the inner product definition of (3.41) the prediction errors of (3.39) can be expressed as the norms of the corresponding filter polynomials:

$$\alpha_m = \langle A_m(z), A_m(z) \rangle_{C^TC} = ||A_m(z)||^2_{C^TC} \qquad 3.43a$$

and

$$\beta_m = \langle B_m(z), B_m(z) \rangle_{C^TC} = ||B_m(z)||^2_{C^TC} \qquad 3.43b$$

Minimizing prediction errors can therefore equivalently be described as minimizing weighted norm squares of the polynomials $A_m(z)$ and $B_m(z)$ for m=1,2,...,M. From the definition of the inner product we see that a filter $A_m(z)$ has a zero norm if the output samples are zero for k=M

through k=L. This event will happen for finite L if the input data exactly satisfies a homogeneous difference equation, whose characteristic polynomial is $A_m(z)$. The norms used in (3.43) are therefore really pseudo norms, and it is related to this fact that we cannot give a one dimensional spectral error criterion for the covariance sequence prediction error minimization.

If $A_m(z)$ minimizes α_m, adding cz^{-j}, j=1,2,..., m to the polynomial must result in a larger norm square:

$$||A_m(z) + cz^{-j}||^2_{C^TC} \geq ||A_m(z)||^2_{C^TC} \quad \text{for } j=1,2,\ldots,m \qquad 3.44a$$

or

$$2c\,\langle A_m(z), z^{-j}\rangle_{C^TC} + c^2\,\langle z^{-j}, z^{-j}\rangle_{C^TC} \geq 0 \quad \forall\, c \qquad 3.44b$$

If $\langle z^{-j}, z^{-j}\rangle$ is non-zero, choosing c as

$$c = -\langle A_m(z), z^{-j}\rangle_{C^TC} / \langle z^{-j}, z^{-j}\rangle_{C^TC} \qquad 3.45$$

leads to

$$-[\langle A_m(z), z^{-j}\rangle_{C^TC}]^2 \geq 0 \qquad 3.46$$

If $\langle z^{-j}, z^{-j}\rangle$ equals zero, then $c = -\langle A_m(z), z^{-j}\rangle_{C^TC}$ leads to the same result (3.46), which must hold for j=1,2,...,m. A necessary condition for minimizing the total squared prediction error therefore is

$$\langle A_m(z), z^{-j}\rangle_{C^TC} = 0 \quad ; \quad j=1,2,\ldots,m \qquad 3.47$$

Similarly derived is the result

$$\langle B_m(z), z^{-j}\rangle_{C^TC} = 0 \quad ; \quad j=1,2,\ldots,m \qquad 3.48$$

Adding an orthogonal polynomial, say Q(z), shows that the orthogonality conditions (3.47) and (3.48) respectively, are also sufficient:

$$||A_m(z) + Q(z)||^2_{C^TC} = ||A_m(z)||^2_{C^TC} + ||Q(z)||^2_{C^TC}$$

$$\geq ||A_m(z)||^2_{C^TC} = \alpha_m \qquad 3.49$$

Therefore the orthogonality conditions (3.47) are necessary and sufficient conditions for minimization of the total squared forward covariance sequence prediction errors.

The orthogonality conditions form the basis for solving for $A_m(z)$ and $B_m(z)$ in a recursive manner. Suppose $A_m(z)$ and $B_m(z)$ have been obtained. Then a linear combination of the form $A_m(z) + k_{m+1}B_m(z)$ will be a polynomial of proper order m+1, with leading coefficient one, and will be orthogonal to the powers $z^{-1},z^{-2},\ldots,z^{-m}$ due to the orthogonality conditions. We therefore have to choose k_{m+1} so as to make the linear combination orthogonal to $z^{-(m+1)}$, and $A_{m+1}(z)$ will have been obtained, and the forward prediction error α_{m+1} will have been minimized. As suggested we will therefore choose

$$A_{m+1}(z) = A_m(z) + k_{m+1}B_m(z) \qquad 3.50$$

and invoke the orthogonality conditions to yield an expression for k_{m+1}:

$$\langle A_{m+1}(z), z^{-(m+1)} \rangle_{C^TC} = 0$$

$$= \langle A_m(z), z^{-(m+1)} \rangle_{C^TC} + k_{m+1} \langle B_m(z), z^{-(m+1)} \rangle_{C^TC} \qquad 3.51$$

Since $A_m(z)$ and $B_m(z)$ are both orthogonal to $z^{-1},\ldots,z^{-m}$ we have

$$\langle A_m(z), z^{-(m+1)} \rangle_{C^TC} = \langle A_m(z), B_m(z) \rangle_{C^TC} = \langle 1, B_m(z) \rangle_{C^TC} \qquad 3.52$$

and also

$$\langle B_m(z), z^{-(m+1)} \rangle_{C^TC} = \langle B_m(z), B_m(z) \rangle_{C^TC} = ||B_m(z)||^2_{C^TC} = \beta_m \qquad 3.53$$

Note that expressions (3.52) and (3.53) illustrate that many inner product calculations which might involve double summations simplify to single summation due to the orthogonality conditions. Substitution of (3.53) into (3.51) yields the choice for k_{m+1} in (3.50)

$$k_{m+1} = -\frac{1}{\beta_m} \langle A_m(z), z^{-(m+1)} \rangle_{C^T C}$$

$$= -\frac{1}{\beta_m} \sum_{i=0}^{m} \{C^T C\}_{m+1,i} \, a_{mi} \qquad 3.54$$

Recall that $B_m(z)$ is a linear combination of powers of z from z^{-1} to $z^{-(m+1)}$. Therefore each polynomial $B_m(z)$ is orthogonal to all lower order polynomials $B_{m-i}(z)$ due to the orthogonality conditions. The polynomials $\{B_m(z)\}$ thus form an orthogonal set, i.e.

$$\langle B_m(z), B_i(z) \rangle_{C^T C} = 0 \quad \text{for } m \neq i \qquad 3.55$$

The latter property together with the orthogonality relations allows a recursive calculation of the total squared forward prediction error in each step of the recursive process. Start with $A_0(z)=1$ and use (3.50). Then

$$A_m(z) = 1 + \sum_{i=1}^{m} k_i \, B_{i-1}(z) \quad ; \quad m > 0 \qquad 3.56$$

With (3.53) and (3.55)

$$||A_m(z)-1||_{C^T C} = \sum_{i=1}^{m} k_i^2 \, \beta_{i-1}$$

$$= ||A_m(z)||^2_{C^T C} - 2 \langle A_m(z), 1 \rangle_{C^T C} + 1$$

$$= 1 - \alpha_m \qquad 3.57$$

The recursion is then given by

$$\alpha_m = \alpha_{m-1} - k_m^2 \, \beta_{m-1} \qquad 3.58$$

Let us initialize the process with

$$A_0(z) = 1 \quad ; \quad B_0(z) = z^{-1} \tag{3.59}$$

Then the initial total squared prediction errors are given by

$$\alpha_0 = \langle A_0(z), A_0(z) \rangle_{C^TC} = \{C^TC\}_{0,0} \tag{3.60a}$$

$$\beta_0 = \langle B_0(z), B_0(z) \rangle_{C^TC} = \{C^TC\}_{1,1} \tag{3.60b}$$

Now compute

$$\langle A_0(z), z^{-1} \rangle_{C^TC} \overset{(3.59)}{=} \langle A_0(z), B_0(z) \rangle_{C^TC}$$

$$\overset{(3.40)}{=} \langle 1, z^{-1} \rangle_{C^TC}$$

$$\overset{(3.42)}{=} \{C^TC\}_{0,1} = \{C^TC\}_{1,0} \tag{3.61}$$

Substituting the latter result in (3.54) yields

$$k_1 = -\frac{\{C^TC\}_{1,0}}{\beta_0} = -\frac{\{C^TC\}_{0,1}}{\beta_0} \tag{3.62}$$

With the recursive relation (3.50) the next order $A_1(z)$ is found so that

$$a_{10} = 1 \quad ; \quad a_{11} = k_1 \tag{3.63}$$

The reduced total squared prediction error is found from (3.58):

$$\alpha_1 = \alpha_0 - k_1^2 \beta_0 \tag{3.64}$$

Next we will develop recursions for obtaining $A_m(z)$ for m=1,2,...,M. At completion, the inverse filter and total squared prediction error are given by

$$A(z) = A_M(z)$$

$$\alpha = \alpha_M \tag{3.65}$$

Assume step m-1 has been completed, so that $B_i(z)$ and β_i for i=0,...,m-1 are known, as well as $A_m(z)$ and α_m. To complete the next step it is

necessary to know $B_m(z)$. Recall that the $B_m(z)$ polynomials form an orthogonal set, so that the classical Gram-Schmidt orthogonalization procedure can be used recursively:

$$B_m(z) = z^{-(m+1)} - \sum_{i=0}^{m-1} \gamma_{mi} B_i(z) \qquad 3.66$$

Then

$$\langle B_i(z), B_m(z) \rangle_{C^TC} = 0 = \langle z^{-(m+1)}, B_i(z) \rangle_{C^TC} - \gamma_{mi}\beta_i \qquad 3.67$$

and therefore

$$\gamma_{mi} = \begin{cases} \langle z^{-(m+1)}, B_i(z) \rangle_{C^TC} / \beta_i \text{ for } \beta_i \neq 0 \\ \text{arbitrary for } \beta_i = 0 \end{cases} \qquad 3.68$$

If $\beta_i \neq 0$ for $i=0,1,2,\ldots,m-1$ then we can write

$$\gamma_{mi} = \frac{1}{\beta_i} \langle z^{-(m+1)}, \sum_{j=1}^{i+1} b_{ij} z^{-j} \rangle_{C^TC}$$

$$= \frac{1}{\beta_i} \sum_{j=1}^{i+1} \{C^TC\}_{m+1,j} b_{ij} \quad \text{for } i=0,1,\ldots,m-1 \qquad 3.69$$

Substituting the latter in (3.66) yields for the coefficients of polynomial $B_m(z)$:

$$b_{m,m+1} = 1$$

$$b_{m,j} = -\sum_{i=0}^{m-1} \gamma_{mi} b_{ij} = -\sum_{i=j-1}^{m-1} \gamma_{mi} b_{ij} \qquad 3.70$$

The total backward prediction error can now be evaluated:

$$\beta_m = \langle B_m(z), B_m(z) \rangle_{C^TC} \overset{3.53}{=} \langle z^{-(m+1)}, B_m(z) \rangle_{C^TC}$$

$$\overset{3.40}{=} \langle z^{-(m+1)}, \sum_{j=1}^{m+1} b_{mj} z^{-j} \rangle_{C^TC}$$

$$\beta_m = \sum_{j=1}^{m+1} b_{mj} \{C^T C\}_{m+1,j} \qquad 3.71$$

Evaluation of $B_m(z)$ and β_m are herewith complete at step m, and $A_{m+1}(z)$ can be evaluated from (3.50) and (3.54). The resulting $(m+1)^{st}$ order forward predictor coefficients are:

$$\begin{aligned} a_{m+1,0} &= 1 \\ a_{m+1,i} &= a_{m,i} + k_{m+1}\, b_{mi} \quad ; \quad i=1,\ldots,m \\ a_{m+1,m+1} &= k_{m+1} \end{aligned} \qquad 3.72$$

The associated covariance sequence prediction error is evaluated from (3.58)

$$\alpha_{m+1} = \alpha_m - k_{m+1}^2\, \beta_m \qquad 3.73$$

This completes a full circle in the iterative process.

3.4 Singularity and Stability Considerations

A Fortran routine implementing the recursive procedures outlined in the previous section, is given in [MG]. These subroutines were used in the simulations performed in the application chapters of this thesis. If during the iterative process a norm squared error equals zero, or becomes negative, the iteration can be terminated since the result is either exceeding the accuracy of the computer, or in error. A zero norm can only occur if the data can be described exactly by a linear combination of complex exponentials, and this would indicate that $C^T C$ in (3.23) is singular. Writing out (3.55) according to the definition gives

$$\langle B_m(z), B_n(z) \rangle_{C^T C} = \sum_{i=1}^{m+1} \sum_{j=1}^{n+1} b_{mi} \{C^T C\}_{i,j}\, b_{nj} = \delta_{mn}\, \beta_n \qquad 3.74$$

In matrix form

$$B^T C^T C\, B = D(\underline{\beta}) \tag{3.75}$$

where

$$B = \begin{bmatrix} 1 & b_{11} & b_{21} & \cdots & b_{M-1,1} \\ & 1 & b_{22} & & b_{M-1,2} \\ & & \ddots & & \vdots \\ & 0 & & \ddots & \vdots \\ & & & & 1 \end{bmatrix} \; ; \; D(\underline{\beta}) = \begin{bmatrix} \beta_0 & & & \\ & \beta_1 & & 0 \\ & & \ddots & \\ & 0 & & \beta_{M-1} \end{bmatrix} \tag{3.76}$$

Since B is a triangular matrix with 1's on the diagonal we derive from (3.75)

$$|C^T C| = |D(\underline{\beta})| = \prod_{m=0}^{M-1} \beta_m \tag{3.77}$$

So when $\beta_n = 0$ for some n, the sequence $\{C_k\}_0^L$ satisfies a homogeneous difference equation of order n+1. We point out once again the importance of the recursive procedure in that it detects singularity, or for that matter, near singularity, of the matrix $C^T C$. During the recursion one may get some idea about the stability of the polynomial $A_m(z)$ as can be deduced from (3.50):

$$\begin{aligned} A_m(z) &= A_{m-1}(z) + k_m B_{m-1}(z) \\ &= \prod_{\ell=1}^{m} (1 - p_{m\ell} z^{-1}) \end{aligned} \tag{3.78}$$

The coefficient with the highest power of z^{-1} yields:

$$k_m = \prod_{\ell=1}^{m} (-p_{m\ell}) \tag{3.79}$$

Therefore, if $|k_m| > 1$, then at least one of the poles $\{p_{m\ell}\}_{\ell=1}^{m}$ has radius larger than one, and the polynomial $A_m(z)$ represents an unstable

system. Alternatively, we can state that a necessary condition for the stability of $A_m(z)$ is:

$$|k_m| < 1 \qquad 3.80$$

The coefficients k_m above relate to both polynomials $A_m(z)$ and $B_m(z)$ however, and are therefore not sufficient to determine $A_m(z)$. One can find a similar but uniquely related set of coefficients κ_m when the polynomials $B_m(z)$ are related to the polynomials $A_m(z)$ by

$$B_m(z) = z^{-(m+1)} A_m(1/z) \qquad 3.81$$

The latter equality is valid when $\{C^TC\}_{ij}$ is a function of (i-j) only, as in the autocorrelation method of linear prediction, under the assumption of stationarity. In making a similar definition however we can derive a set of reflection coefficients $\{\kappa_i\}_1^m$ that uniquely relates to the polynomial $A_m(z)$. Let us define

$$Q_m(z) = z^{-(m+1)} A_m(1/z) \neq B_m(z) \qquad 3.82$$

Combination of (3.82) and (3.78) yields

$$Q_m(z) = z^{-1}[\kappa_m A_{m-1}(z) + Q_{m-1}(z)] \qquad 3.83a$$

or

$$Q_{m-1}(z) = zQ_m(z) - \kappa_m A_{m-1}(z) \qquad 3.83b$$

Substituting in (3.78) and solving for $A_{m-1}(z)$ in terms of $A_m(z)$ gives:

$$A_{m-1}(z) = \frac{A_m(z)-z\kappa_m Q_m(z)}{1-\kappa_m^2} \quad ; \quad |\kappa_m| \neq 1 \qquad 3.84a$$

With (3.82)

$$A_{m-1}(z) = \frac{A_m(z)-z^{-m}\kappa_m A_m(1/z)}{1-\kappa_m^2} \quad ; \quad |\kappa_m| \neq 1 \qquad 3.84b$$

or

$$\sum_{i=0}^{m-1} a_{m-1,i} z^{-i} = \frac{\sum_{i=0}^{m} a_{mi} z^{-i} - \kappa_m \sum_{i=0}^{m} a_{m,m-i} z^{-i}}{1-\kappa_m^2} \quad ; \quad |\kappa_m| \neq 1 \qquad 3.84c$$

From the higher order polynomial $A_m(z)$ and κ_m we can therefore recursively find $A_{m-1}(z)$:

$$a_{m-1,i} = \frac{a_{mi} - \kappa_m a_{m,m-i}}{1-\kappa_m^2} \quad ; \quad i=0,1\ldots,m-1 \; ; \; |\kappa_m| \neq 1 \qquad 3.85$$

and thereby κ_{m-i}:

$$\kappa_{m-1} = a_{m-1,m-1} \qquad 3.86$$

The latter procedure provides an efficient recursive procedure to test the magnitude of the reflection coefficients κ_m, corresponding uniquely to the filter polynomial $A_m(z)$, at every step of the recursive process to find the optimal linear covariance sequence predictor. A necessary and sufficient condition for stability of the polynomial $A_m(z)$ is [MG]

$$|\kappa_m| < 1 \quad \forall\, m \qquad 3.87$$

In this section a recursive procedure has been given to solve for the best linear covariance sequence predictor (3.23). The order of the predictor polynomial is increased in every step. Efficient procedures have been given to evaluate the stability of the corresponding inverse filter as well as the residual total squared covariance sequence prediction error for that step. One can therefore stop increasing the order when instability is indicated and say that $A_{m-1}(z)$ is the best linear predictor of maximum order that has a stable inverse. Also, whenever β_m is numerically equal to zero we have arrived at the best linear predictor of minimum order. The latter property enables us to

find the desired solution even if C^TC in (3.23) is singular. Note that we assumed the maximum order of the underlying system to be M. Such an assumption requires an educated guess or a restriction due to implementation requirements. In the above procedures however it is not necessarily a disadvantage to choose the maximum order somewhat too high, as during the recursive process the residual prediction error α_m provides an indication of the resulting performance. The behavior of $\{\alpha_m\}$ thus provides a built-in estimate of the appropriate system order.

Similarities of the present pole identifier with the covariance method of linear prediction of speech are clear. In the latter application the need arises for formant estimation. A damped sinusoidal component of the vocal tract acoustic impulse response is a formant. The analogy with the damped exponentials $\{p_i\}$ in our covariance sequence model of (3.1) is obvious. Formant estimation for speech signals is therefore an analog for pole identification or estimation for covariance sequences. The accuracy of formant estimation obtained with the covariance method of linear prediction has turned out to be vastly superior to the results derived with the more popular autocorrelation method of linear prediction [CL]. In particular when only short data sequences were available this supremacy is pronounced. The desirability of the linear prediction approach for formant estimation is qualitatively justified by the fact that in signal representation, the most efficient basis set (or set of approximating functions) is the one which most closely matches, in some sense, the desired characteristics of the signal. The latter statement reflects remarks made at the closing of Section 5.2. We indicated there that the proposed covariance sequence model will fit certain processes with the minimum number of coefficients

because of its underlying structure. In the exposition for the design of a certain class of recursive digital filters this philosophy will play an important role. See also [GRT].

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4 COVARIANCE SEQUENCE APPROXIMATION FOR PARAMETRIC SPECTRUM ANALYSIS

An introduction is given in Section 4.1 that concentrates on parametric spectrum analysis and why it might be beneficial to take such an approach. The section after contains a derivation of the parametric spectral density estimate associated with the covariance sequence approximation procedure of Chapter 3. Its properties and peculiarities are examined as well. Section 4.3 extends the covariance sequence approximant and the associated spectral density estimate so as to include the analysis of harmonic processes in additive noise. Sections 4.4 and 4.5 give results for the covariance sequence approximant applied to known and estimated covariance sequences, respectively. Application of the basic idea to the fitting of a known but nonrational Gaussian spectral density indicates the technique is robust.

4.1 Parametric Spectrum Analysis Introduction

In order to see parametric spectrum analysis procedures in the proper perspective we will briefly review some ideas pertaining also to nonparametric spectrum analysis. Let us first note that our interest will be in spectral estimation for sampled data. For estimation of spectra from continuous time systems this means that the resulting spectral estimate is an infinite sum of translates of the continuous time spectrum. The socalled aliasing effect will be ignorable for a proper choice of the sampling frequency. Exact determination of either covariance function or spectrum would require a perfectly measured, infinitely long realization and infinite wordlength computation for all wide sense stationary processes with nondeterministic components. Both requirements are impractical. Approximate determination however immediately

brings up the questions of how long the observation interval should be, which computational approach should be used and finally how much confidence one can have in the resulting estimate. The answers are relatively simple for nonparametric spectrum estimation [BT], but unsatisfactory because of the required observation length for highly precise estimates. The finite observation interval has the effect that the resulting spectral estimate is at best a convolution of the continuous time spectrum with the Lanczos spectral window corresponding to a finite length observation sequence. The latter has the effect of reducing the spectral resolution. It is imperative therefore that one realizes that a finite length data sequence yields no more than estimates for the covariance function and the spectrum, and that consequently these estimates exhibit bias and variance. In parametric approaches however observation window and spectrum estimate are deconvolved in some sense by implicit extension of the data sequence.

Many methods for spectrum estimation have been proposed over the years, both parametric and nonparametric ones. Methods based on finding a spectrum estimate for the observed data directly have received an impulse due to the computational advantages of the FFT algorithm. The underlying assumption of periodicity was far outweighed by the conveniences of the method. A finite data sequence yields a finite covariance sequence and the corresponding spectrum has finite spectral resolution due to the finite observation window. The resolution properties of the conventional spectral analysis methods are chiefly due to the assumptions that are made concerning the data outside of the observation interval. The desire to be consistent with the available covariance sequence and at the same time be noncommittal about the unobserved data, has led recently to the maximum entropy spectral estimator [B1], also known as

a high resolution spectral estimator. The maximum entropy estimator can be derived from maximum entropy considerations for Gaussian processes, and it was shown later that the corresponding extrapolation of the covariance sequence is equivalent to least squares fitting of an autoregressive model to the process [VDB]. The high resolution is achieved by parametric modeling and implicity extending the available data. The Burg algorithm does not explicitly make an estimate of the covariance sequence lags. Instead it operates directly on the observed data samples and minimizes the sum of forward and backward prediction errors simultaneously [B2]. It has been found that this procedure is quite sensitive to the initial phase of observed sinusoids [CS]. A least squares estimator minimizing the same error criterion as in the Burg algorithm, but based on explicit computation of the corresponding covariance sequence, was found to be much more stable with respect to the initial phase of the signal [UC]. The high resolution spectral estimators have been very successful in resolving pronounced spectral peaks, but a word of caution is not out of place. The notion of high resolution would seem to indicate that all spectral features can be resolved well. It is known however that influential zeros in a spectrum are not modeled very well by the underlying all pole model of any autoregressive spectral estimator. We must therefore recognize that the term high resolution spectral estimation has chiefly pertained to the resolution of spectral peaks. The smooth nature of the corresponding spectrum estimate is due to the fact that it is the spectrum of a process, for which the available data merely represents a realization. Such a process spectrum will of course look much smoother than a spectrum estimate based on the realization directly.

In the parametric spectrum estimation approach of this thesis we pose the basic question of parametric spectrum analysis: Which system, driven by white noise, is likely to have generated the present finite realization? The spectrum associated with that system is called the spectral estimate. For many applications one furthermore wishes to have a low parameter representation for purposes of bandwidth efficient communication as in linear predictive coding of speech [MG]. Maximum entropy spectral analysis and other autoregressive spectrum estimators, lead to a spectral density plot and an AR model, but do not readily give frequency and corresponding power estimates. Reasonable approximations have been suggested for low noise cases with known covariances [JA]. The latter method based on approximating complex residues yielded improved estimates over those based on integration of the spectral density around peaks [LA].

One of the most serious problems in parametric spectrum analysis is the determination of an appropriate order for the underlying model. If in the autoregressive spectrum estimators too few parameters are used a highly smoothed spectral estimate results. If on the other hand too many parameters are determined, spurious peaks will turn up. This order determination problem is most pronounced for estimation from short realizations. In the proposed approach of this thesis spurious poles can turn up in the initial stage of the approximation, but the corresponding linear combination weights provide a correction mechanism to assess their proper importance. For maximum entropy and other autoregressive spectral estimators order determination criteria have been proposed and in many instances used quite successfully [AK], [PA]. In the proposed spectral estimator the order is increased recursively and a simultaneous

indication of the covariance sequence prediction error points out if the order is adequate before reaching a preset maximum order. An error criterion as Akaike's starts to increase at some order and intuitively reflects the danger of using an ever increasing model order due to the fact that with high enough order even extremely noisy data can be fitted exactly. This philosophy led in the system identification and time series modeling area to the Group Method of Data Handling [IV]. In the GMDH method model parameters are based on one data set and model appropriateness is judged from another, preferably disjoint data set.

The assumption of sinusoids in white noise led to the covariance sequence model associated with the Pisarenko decomposition [PI]. Pisarenko points out that the maximum entropy spectrum for harmonic processes in additive white noise is a smoothed version of the Pisarenko decomposition result. As no autoregressive spectral estimator has any zeros in its spectrum, it is obviously the proximity of the zeros to the unit circle that determines the resolution power of spectral estimators for harmonic processes. The latter property indicates the usefulness of ARMA spectral estimators, several of which have been proposed [G2], [KPSS], [K], [TS]. The minimum residual criterion in the latter determines the zeros first with an iterative Powell minimization technique and thereafter performs a further whitening procedure on the filtered signal resulting in a linear system of equations for finding the poles. When the actual signal spectrum has influential zeros, quite a few poles are needed in an all pole model to achieve a good approximation. The introduction of zeros will then certainly allow a more efficient representation.

Part of the significance of ARMA modeling of course lies in the fact that sampling of a continuous AR process of order M results in a discrete ARMA(M,M-1) [BA]. That property holds for any strictly proper continuous time process of order M [P]. Furthermore, any continuous spectral density function can be approximated arbitrarily closely by an ARMA(M,N) process, by proper choice of the parameters. While this is also true of spectral densities of finite MA and finite AR processes, a model with a rational spectral density will almost invariably lead to a better fit with fewer parameters [K]. The latter property has been referred to as one of parsimony [BJ].

4.2 Parametric Spectrum Estimation Based on Covariance Sequence Approximation

Recall that based on a finite covariance sequence sample $\{C_k\}_{-L}^{L}$ we find an approximating sequence

$$R_k = \sum_{i=1}^{M} A_i \, p_i^{|k|} \qquad \forall k \tag{4.1}$$

where $\{A_i, p_i\}$ are real or occur in complex conjugate pairs. Since the spectrum is uniquely related to the covariance sequence, we may find the approximating spectrum as follows

$$\begin{aligned} \hat{S}(z=e^{j\theta}) &= \mathcal{Z}\{R_k\} = \sum_{-\infty}^{\infty} R_k \, z^{-k} \\ &= \sum_{i=1}^{M} A_i \sum_{-\infty}^{\infty} p_i^{|k|} \, z^{-k} \\ &= \sum_{i=1}^{M} A_i \frac{(1-p_i)(1+p_i)}{(z-p_i)(z^{-1}-p_i)} \quad ; \quad -\pi \leq \theta < \pi \end{aligned} \tag{4.2}$$

The region of absolute convergence is the annulus determined by the radii of $\max_i |p_i|$ and its inverse. The approximating spectrum does not

exist as a uniform limit if any of the complex exponentials has radius greater than or equal to one. This situation would correspond to a non-minimum phase covariance sequence prediction filter. As explained in the previous chapter this can be prevented from happening in the recursive determination of the filter. We further note here that the convergence of $\hat{S}(z)$ has to be examined separately if the maximum pole radius equals one.

Let us examine some properties of this spectrum estimate. From (4.2) it can be written as follows

$$\hat{S}(z) = \frac{N(z)}{\prod_{i=1}^{M} (z-p_i)(z^{-1}-p_i)}$$

$$= \frac{N(z)}{A(z)A(z^{-1})} \qquad 4.3a$$

where $A(z)$ is the denominator polynomial of the underlying ARMA$(M,\bar{M})$ system. The numerator $N(z)$ is given by:

$$N(z) = \sum_{i=1}^{M} A_i(1-p_i)(1+p_i) \prod_{\substack{k=1 \\ \neq i}}^{M} (z-p_k)(z^{-1}-p_k) \qquad 4.3b$$

Obviously $N(z) = N(z^{-1})$, so if z_o is a root of $N(z)$, then $1/z_o$ is also a root. Let us define the following polynomial [K]

$$\phi(z) = z^{(M-1)} N(z) \qquad 4.4$$

this polynomial is of degree $2(M-1)$ in z. Again if z_o is a root then $1/z_o$ is a root, and the polynominal can therefore be written

$$\phi(z) = \text{Const} \prod_{j=1}^{M-1} (z-z_j)(zz_j-1) \qquad 4.5$$

From (4.4)

$$N(z) = z^{-(M-1)} \phi(z)$$

$$= \text{Const} \prod_{j=1}^{M-1} (z-z_j)(z^{-1}-z_j)$$

$$= B(z)B(z^{-1}) \qquad 4.6$$

B(z) contains one root of each of the pairs $(z_j, 1/z_j)$ and the Const. depends on the choice of roots. For a given polynominal N(z) we can find several polynominals B(z), for each of which the approximate spectrum can be written in factored form:

$$\hat{S}(z) = \frac{B(z)B(z^{-1})}{A(z)A(z^{-1})} \qquad 4.7$$

As a result we say that there exist several ARMA(M,$\bar{M}$) systems with transfer function

$$H_{ARMA}(z) = \frac{B(z)}{A(z)} \qquad 4.8$$

and spectrum $\hat{S}(z)$ provided B(z) has real coefficients, The denominator polynomial has as its coefficients the predictor polynominal coefficients identified in the previous chapter and they are all real. Also from (4.3b) it is clear that N(z) in a mirror image polynomial with real coefficients due to the fact that $\{A_i, p_i\}$ are real or occur in complex conjugate pairs. For N(z) we then have the following implications

$$N(z_o) = 0 \rightarrow N(z_o^*) = 0 \qquad 4.9a$$

$$N(z_o) = 0 \rightarrow N(z_o^{-1}) = 0 \qquad 4.9b$$

A zero inside the unit circle, say $z_o = \rho e^{j\theta_o}$; $0 < \rho < 1$, will therefore mean that the following four zeros exist

$$\{\rho e^{j\theta_o}, \rho e^{-j\theta_o}, \frac{1}{\rho} e^{-j\theta_o}, \frac{1}{\rho} e^{j\theta_o}\} \qquad 4.10$$

We have seen that B(z) in (4.6) contained one root of each of the pairs $(z_j, 1/z_j)$, and we can easily see now that choosing $\rho e^{j\theta_o}$ and $\frac{1}{\rho} e^{j\theta_o}$ will not lead to an ARMA numerator polynomial B(z) with real coefficients. Therefore, if we desire an ARMA system with real coefficients it is imperative to choose the zeros in complex conjugate pairs. Of the zeros in (4.10) only one free choice can be made. If a zero of N(z) occurs

inside the unit circle and is real, another zero will occur outside the unit circle, with inverse radius. The necessary conditions (4.9) are satisfied even when only one extra zero occurs. Either one of these real roots can be chosen for the numerator polynomial B(z).

Let us now investigate the occurrence of a non-repeated zero on the unit circle: $z_o = e^{j\theta_o}$. In total compliance with (4.9) we have as the only other zero $z_o^* = e^{-j\theta_o}$. Again either one of these roots can be chosen for the polynomial B(z). This complex zero does not occur in a complex conjugate pair however, and the resulting ARMA system has complex numerator coefficients. For B(z) to have real coefficients the zeros of N(z) on the unit circle must occur in multiples of two. The latter requirement turns up invariably in spectral factorization problems [VT]. Violation of this requirement indicates that the corresponding spectrum estimate is negative for some frequencies. The conditions of (4.9) which are satisfied in the spectrum estimate of (4.2), cannot guarantee the spectrum estimate to be nonnegative. Equivalently, the covariance sequence approximant is not necessarily nonnegative definite, in which case it is not a covariance sequence. It should be noted that negative spectrum estimates are not uncommon, and may occur in particular when the observations do not exactly satisfy the underlying model. One can then take the spectral estimate to be zero [G1], or alternatively plot absolute value of $\hat{S}$. If the interest is purely in a spectral estimate defining the estimate to be zero where it is negative seems at least a reasonable thing to do. If there is interest in the complex ARMA(M,$\bar{M}$) associated with the spectral estimate, then taking the absolute value is more appropriate.

Since the denominator polynomial A(z) has real coefficients one can find the corresponding complex linear weights A_i in the model of (4.1)

under the condition that the resulting covariance approximant be nonnegative definite or equivalently the spectrum estimate be nonnegative. It is fairly easy to give sufficient conditions, such as

$$A_i \text{ real, } > 0 \quad ; \quad i = 1,\ldots,M \tag{4.11}$$

Even a substantially less restrictive condition can be given by direct application of a continuous time restriction [Y]:

$$|\psi_i| \leq \arctan \frac{\alpha_i}{\beta_i} \tag{4.12}$$

$$A_i = r_i e^{-j\psi_i} \quad ; \quad p_i = \rho_i e^{j\beta_i}$$

$$\alpha_i = - \ell n \, \rho_i$$

The above restrictions would assure the nonnegative definiteness of each of the individual covariance sequences corresponding to a complex conjugate pole pair. It is very difficult to derive necessary and sufficient conditions for the general model of (4.1) to result in a nonnegative spectrum estimate. Imposition of these conditions would immediately result in a nonlinear programming problem, which is what we are trying to avoid.

4.3 Harmonic Processes in Additive Noise

An important class of processes is formed by a linear combination of random amplitude, random phase sinusoids in additive noise. In case the additive noise is white this is exactly the model that underlies Pisarenko's method of spectral decomposition [PI]. Strictly speaking one cannot define the spectrum for a random amplitude, random phase sinusoid. Consequently, unless one proceeds with some care, the spectrum of a random amplitude, random phase sinusoid has no well-defined meaning. Therefore, let us proceed as follows. Define the process

$$X_k = \alpha \sin(k\theta_1 + \phi) \quad ; \quad k = 0, \pm 1,\ldots \tag{4.13}$$

with α: $N(0,\sigma^2)$, say, and ϕ: $U[0,2\pi]$. The covariance of $\{X_k\}$ is

$$C'_k = \sigma^2 \cos k\theta_1 \quad ; \quad k = 0, \pm 1, \ldots \qquad 4.14$$

This covariance sequence is neither ℓ_1 nor ℓ_2, so for the time being the formal spectrum with double poles at $z = e^{\pm j\theta_1}$,

$$S_1(z=e^{j\theta}) = \frac{\sigma^2}{2} \frac{(1-e^{j\theta_1})(1+e^{j\theta_1})}{(e^{j\theta}-e^{j\theta_1})(e^{-j\theta}-e^{j\theta_1})} + \frac{\sigma^2}{2} \frac{(1-e^{-j\theta_1})(1+e^{-j\theta_1})}{(e^{j\theta}-e^{-j\theta_1})(e^{-j\theta}-e^{-j\theta_1})} \qquad 4.15$$

has only formal meaning. To give it a well defined sense, and to show how it may be obtained as a limiting case of the spectrum given in (4.2), define the covariance sequence

$$R_k^\rho = \rho^{|k|} \sigma^2 \cos k\theta_1 \; ; \qquad 0 < \rho < 1 \qquad 4.16$$

The corresponding spectrum, with poles at $z = \rho e^{\pm j\theta_1}$, is well defined. From (4.2)

$$S_\rho(z=e^{j\theta}) = \frac{\sigma^2}{2} \left[\frac{(1-\rho e^{j\theta_1})(1+\rho e^{j\theta_1})}{(e^{j\theta}-\rho e^{j\theta_1})(e^{-j\theta}-\rho e^{j\theta_1})} + \frac{(1-\rho e^{-j\theta_1})(1+\rho e^{-j\theta_1})}{(e^{j\theta}-\rho e^{-j\theta_1})(e^{-j\theta}-\rho e^{-j\theta_1})} \right] \qquad 4.17$$

Furthermore, $S_\rho(z=e^{j\theta})$ may be written [LLS], [F]

$$S_\rho(z=e^{j\theta}) = \int_{-\pi}^{\pi} P_\rho(\theta-\phi) S_1(z=e^{j\phi}) d\phi \qquad 4.18a$$

Here $P_\rho(\theta)$ is the Poisson kernel:

$$P_\rho(\theta) = \frac{1-\rho^2}{2\pi(1-2\rho\cos\theta + \rho^2)} \qquad 4.18b$$

By the delta function character of the Poisson kernel sequence, we have $S_\rho(z=e^{j\theta})$ converging to $S_1(z=e^{j\theta})$ at all points of continuity [L]:

$$S_\rho(z=e^{j\theta}) \xrightarrow[\rho \uparrow 1]{} S_1(z=e^{j\theta}) \qquad 4.19$$

Thus the formal spectral density $S_1(\cdot)$ for a random amplitude, random phase sinusoid may be obtained as a uniform limit (as $\rho \uparrow 1$) of the spectra defined by our covariance sequence approximant. The above procedure generalizes to a linear combination of random amplitude, random phase sinusoids in an obvious way, by the independence of the amplitude and phase variables. A mix of discrete and continuous spectral density functions can now be represented by the covariance sequence approximant of (4.1) by virtue of the independence of the discrete and continuous components as in the Wold decomposition theorem.

Due to symmetry properties we are mainly concerned with the positive going part of the covariance sequence. The covariance sequence approximant for nonnegative lags is easily recognized as the general solution of a linear homogeneous difference equation of order M with real coefficients and a characteristic equation with simple roots. The first M lag values are determined by the initial conditions and for lag values M and up the behavior of the solution is determined by the characteristic roots. Viewing the covariance sequence approximant as the solution of a linear homogeneous difference equation we readily recognize its validity for all characteristic roots, whether in, on or outside of the unit circle.

In order to show how white noise can be represented by the covariance sequence approximant of (4.1) let us assume a first order autoregressive process driven by white noise ε: (0,1). Assume further that the autoregressive coefficient is positive real. The resulting noise process η_k is described by

$$\eta_k = \rho\, \eta_{k-1} + \varepsilon_k \qquad 4.20$$

Multiplying by r_{k-i-1} and taking expected values, a recursive relation for the noise covariance sequence is found

$$C^{\rho}_{i+1} - \rho\, C^{\rho}_{i} = 0 \qquad \forall_i > 0 \qquad 4.21$$

Consequently we can find the positive going solution to (4.21) and extend it for negative lag values by the symmetry property to yield the explicit expression for the noise covariance sequence:

$$C^{\rho}_{k} = \frac{\sigma_\varepsilon^2}{1-\rho^2}\ \rho^{|k|} \qquad \forall k \qquad 4.22$$

Note that if we choose $0 < \rho < 1$ then the sequence of sequences $\{C^{\rho}_{k}\}_{-\infty}^{\infty}$, indexed by ρ, converges in ℓ_1 to the dirac delta sequence $\{\sigma_\varepsilon^2\delta(k)\}_{-\infty}^{\infty}$, as ρ approaches zero. This uniform convergence furthermore implies that the spectrum $S^{\rho}(\theta)$ associated with (4.22) converges uniformly and absolutely to 1 for $\theta\varepsilon[-\pi,\pi]$. With the radius of the noisepole ρ approaching zero we therefore have an excellent approximation to a white noise process. As the covariance sequence in (4.22) is seen to be a special case of the general covariance sequence approximant of (4.1) with A_i positive real, we can state that the covariance sequence for the white noise process is a limiting case of the covariance sequence approximant. Applying the limiting arguments for sinusoids and white noise simultaneously, Pisarenko's decomposition technique becomes a special case of the covariance sequence approximant proposed in Chapter 2 of this thesis. We furthermore note that both limiting arguments above resulted in positive real linear combinations of complex exponentials so that sinusoids in white noise also form a limiting case for the generalized wide sense stationary process of Section 2.5.

With two complex poles per sinusoid the covariance sequence approximant implicitly models a sinusoid in white noise with an

ARMA(3,2) model. In view of the ARMA model for harmonic processes in white noise derived by Ulrych and Clayton [UC], the present modeling approach is not the most parsimonious one. We note, however, that in the present approach no eigenproblem needs to be solved. Furthermore, in the most parsimonious ARMA(p,p) model, numerator and denominator polynomial are equal, due to the fact that the model describes the process as if the additive noise were its input. The only information in the ARMA(p,p) model therefore concerns the pole locations exclusively. We emphasize again that sinusoids in white noise represent a special case for the covariance sequence approximant. The sinusoids happen to correspond to poles on the unit circle and the white noise corresponds to an autoregressive pole at zero. In the sections to follow we will examine the estimator results for known covariance sequences first, and thereafter for estimated covariance sequences. We will furthermore experimentally test the robustness of the covariance sequence approximant against nonrational spectra by applying it to the known covariance sequence for the Gaussian spectral density.

4.4 Numerical Results for Known Covariance Sequences

In this section we examine the performance of the covariance sequence approximant, and the corresponding spectral density estimate, when the given finite length covariance sequence samples are known. The resulting covariance sequence approximant is given in numerical form by real and imaginary parts of the linear combination factors A_i and by radius and angle (in radians) for the complex exponentials p_i. Corresponding to the covariance sequence approximant is the parametric spectral density plot of $10 \log_{10}|\hat{S}(z=e^{j\theta})|$; $0 \leq \theta \leq 1$ (in π radians). Recall that absolute value is necessary due to the fact that our "spectral density" estimate may be negative for some frequencies. For

poles with radius within 1_{10}^{-3} of 1 the discrete spectral density is superimposed on the spectral density plot and depicted as a solid vertical line at the frequency θ_i determined for the sinusoid. The maximum value of the spectral line is given by $10 \log_{10} 2|A_i|$, which indicates the power of the sinusoid when A_i is real. In every example a maximum order M was chosen a priori. Unless otherwise indicated poles are determined from lags M and up, whereas linear combination weights are determined from all lags given. All computations were performed in single precision arithmetic on the CYBER 171 at Colorado State University.

Example 1 A single sinusoid

Given $C_k = \cos\pi/4.k$ for lags $k = 0,\ldots,4$

Maximum predictor order: M = 2

Covariance Sequence Approximant:

$$R_k = (.5000 + j\ .4885_{10}^{-14})(1.000 \exp j\ .7854)^{|k|} + \text{c.c.}$$

where c.c. in the sequel denotes complex conjugate of the term in front of it.

The identified frequency is correct to four decimal places. Note also that the power estimate given by R_o is correct to at least four decimal places. The imaginary part of the linear weights is truly negligible as compared to the real part, which is to be expected in view of the representation in (4.19). See Figure 4.1 for the spectral plot.

Example 2 A single sinusoid in white noise

A. Given $C_k = \cos\pi/4.k + .0001\ \delta(k)$ for lags $k = 0,\ldots,6$ (SNR=+40dB)

Maximum predictor order: M = 3

Covariance Sequence Approximant:

$$R_k = (.5000 - j\ .5631_{10}^{-10})(1.000 \exp j\ .7854)^{|k|} + \text{c.c.}$$
$$+ (.1000_{10}^{-3})(.5684_{10}^{-5} \exp -j\ 3.142)^{|k|}$$

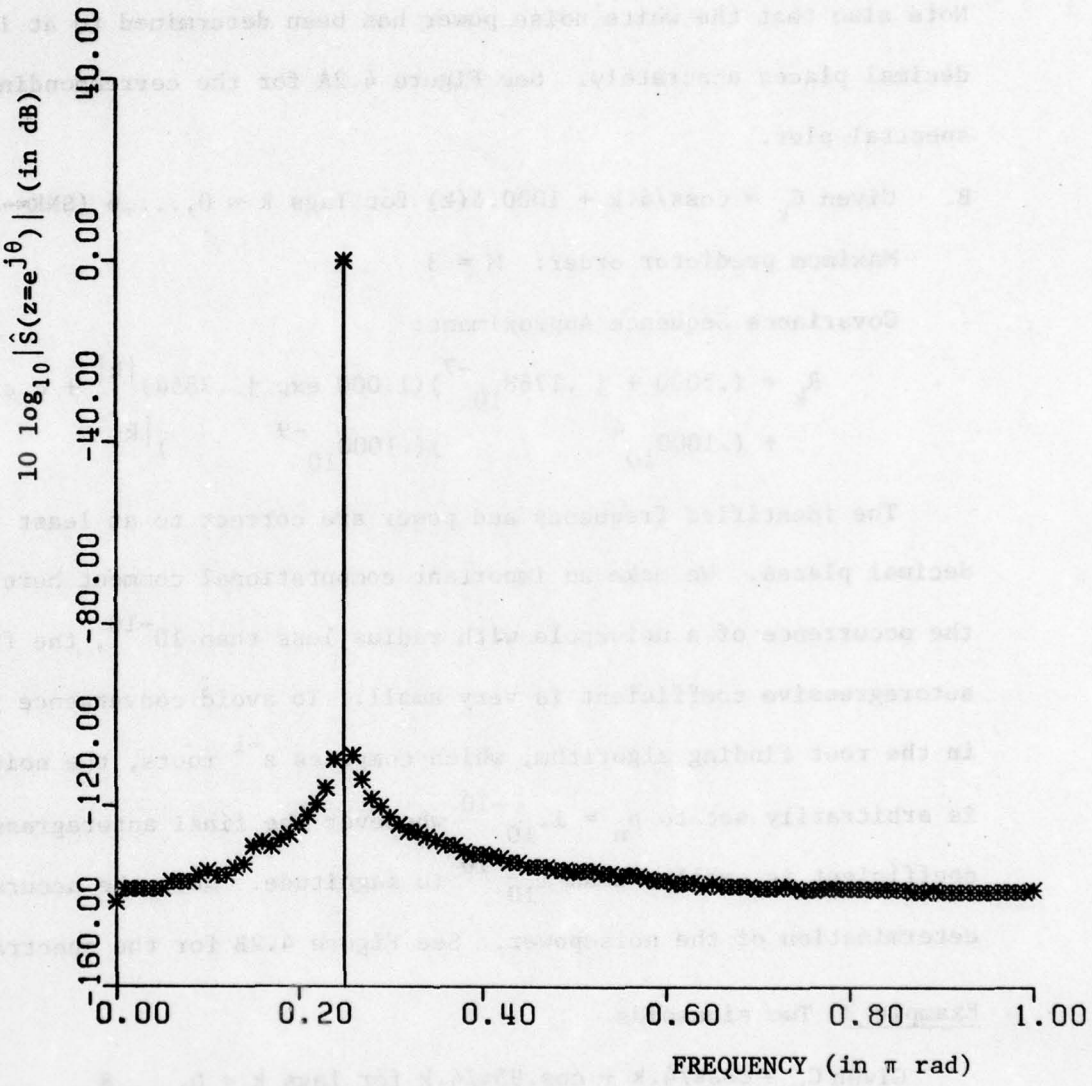

Figure 4.1 Spectral estimate for sine from 5 known covariance lags.

The identified frequency and power are correct to at least four decimal places. Note here that the noisepole results in an excellent first order autoregressive noise approximation to the given white noise. Note also that the white noise power has been determined to at least four decimal places accurately. See Figure 4.2A for the corresponding spectral plot.

B. Given $C_k = \cos\pi/4.k + 1000.\delta(k)$ for lags $k = 0,\ldots,6$ (SNR=-30dB)

Maximum predictor order: M = 3

Covariance Sequence Approximant:

$$R_k = (.5000 + j\ .1768_{10}{}^{-7})(1.000 \exp j\ .7854)^{|k|} + c.c.$$
$$+ (.1000_{10}{}^{4}\qquad\qquad)(.1000_{10}{}^{-9}\qquad)^{|k|}$$

The identified frequency and power are correct to at least four decimal places. We make an important computational comment here. With the occurrence of a noisepole with radius less than 10^{-10}, the final autoregressive coefficient is very small. To avoid convergence problems in the root finding algorithm, which computes z^{-1} roots, the noisepole is arbitrarily set to $p_n = 1._{10}{}^{-10}$ whenever the final autoregressive coefficient is smaller than $1_{10}{}^{-10}$ in magnitude. Note the accurate determination of the noisepower. See Figure 4.2B for the spectral plot.

Example 3 Two sinusoids

Given $C_k = \cos\pi/4.k + \cos.95\pi/4.k$ for lags $k = 0,\ldots,8$

Maximum predictor order: M = 4

Covariance Sequence Approximant:

$$R_k = (.5000 + j\ .3499_{10}{}^{-4})(1.000 \exp j\ .7854)^{|k|} + c.c.$$
$$+ (.5000 - j\ .3499_{10}{}^{-4})(1.000 \exp j\ .7461)^{|k|} + c.c.$$

To 4 decimal accuracy $\pi/4 = .7854$ and $.95\ \pi/4 = .7461$, which implies accurate determination of both closely spaced frequencies.

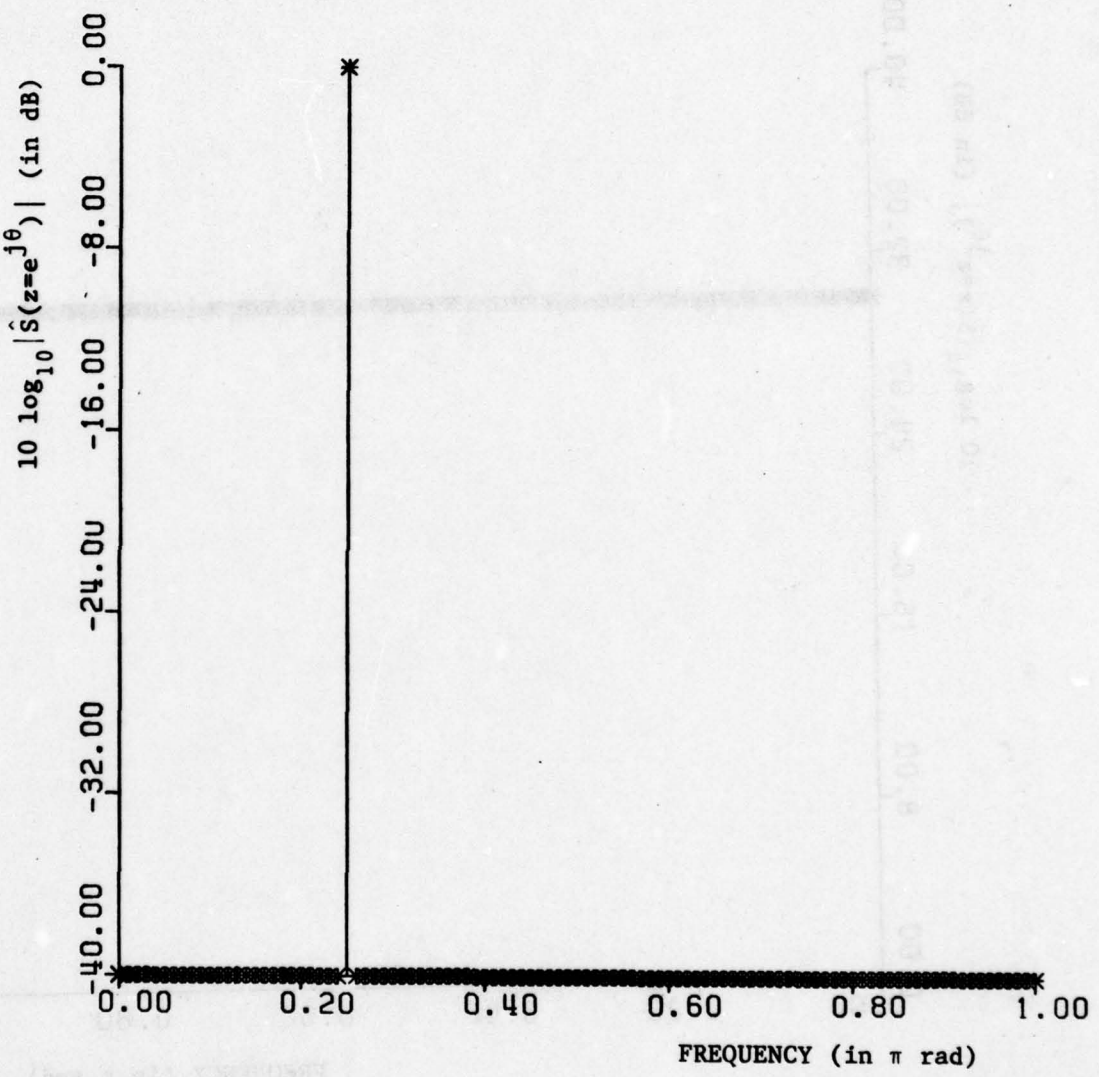

Figure 4.2A Spectral estimate for sine in white noise. Given 7 known covariance lags. SNR = 40 dB.

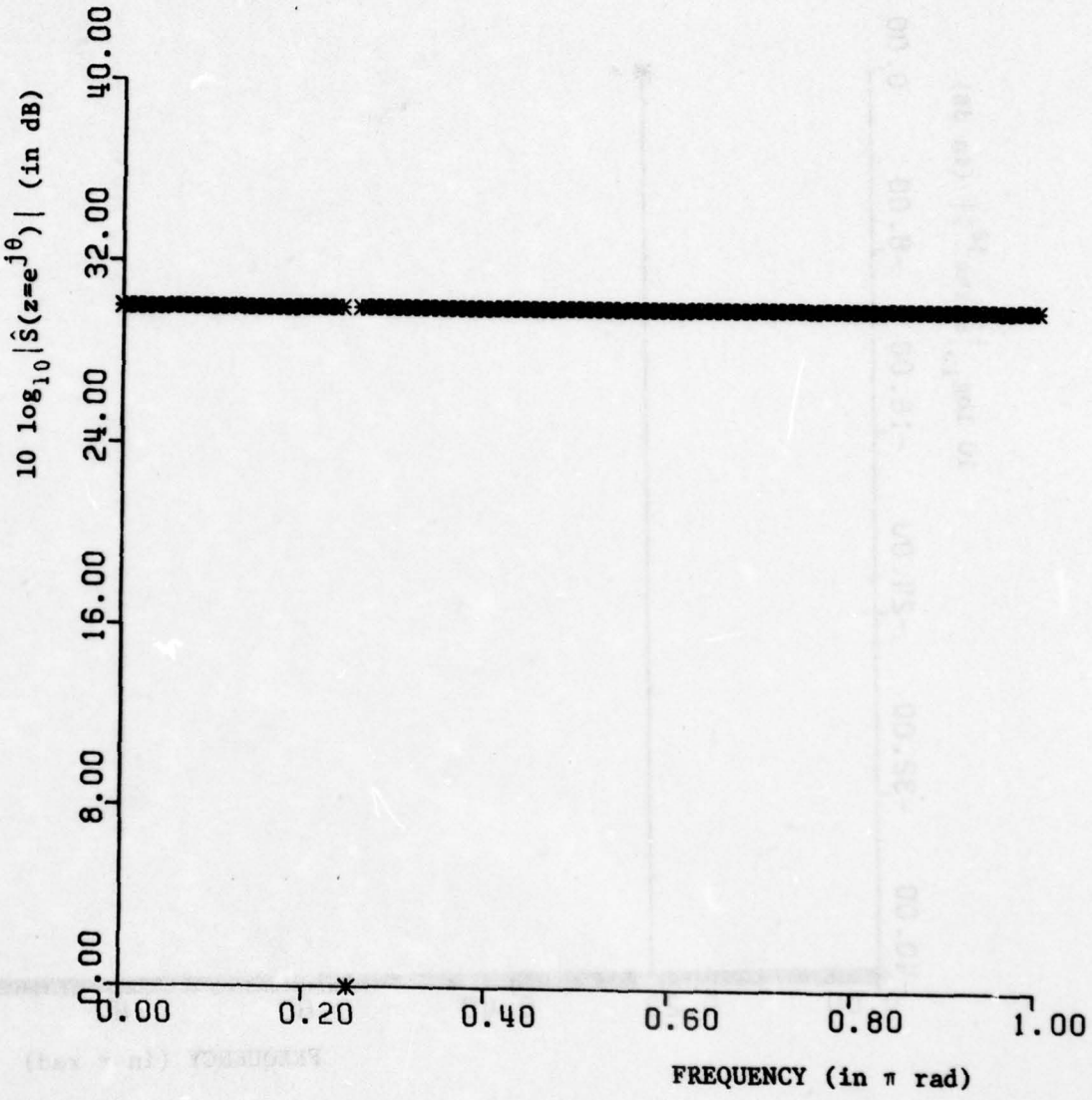

Figure 4.2B Spectral estimate for sine in white noise. Given 7 known covariance lags. SNR = -30 dB.

The corresponding power estimates are equally accurate. See Figure 4.3 for visual spectral information.

Example 4 Two sinusoids in white noise

A. Given $C_k = \cos\pi/4.k + .01 \cos.95\pi/4.k + 100\ \delta(k)$ for lags $k = 0,\ldots,14$ (SNR=-20dB, -40dB)

Maximum predictor order: M = 5

Covariance Sequence Approximant:

$$R_k = (.5000 + j\ .4912_{10}{}^{-5})(1.000 \exp j\ .7854)^{|k|} + c.c.$$
$$+ (.4997_{10}{}^{-2} - j\ .4918_{10}{}^{-5})(1.000 \exp j\ .7461)^{|k|} + c.c.$$
$$+ (.1000_{10}{}^{3})(.1000_{10}{}^{-10})^{|k|}$$

Note both closely spaced frequencies, their quite different powers and the power of the noise they are submerged in, are all identified to at least 3 decimal places. See Figure 4.4A.

B. Given $C_k = \cos\pi/4.k + .001 \cos.95\pi/4.k + 1000.\ \delta(k)$ for lags $k = 0,\ldots,11$ (SNR-30dB, -60dB)

Maximum predictor order: M = 5

Covariance Sequence Approximant:

$$R_k = (.4993 - j\ .5759_{10}{}^{-3})(1.000 \exp j\ .7854)^{|k|} + c.c.$$
$$+ (.1234_{10}{}^{-2} + j\ .5763_{10}{}^{-3})(.9935 \exp j\ .7622)^{|k|} + c.c.$$
$$+ (.1000_{10}{}^{4})(.1033_{10}{}^{-9} \exp j\ 3.142)^{|k|}$$

Single precision computation starts to affect the least powerful component here. Note that the relative accuracy is determined by the relative importance of each component in terms of power. For visual spectral information see Figure 4.4B, and note that frequency .95 $\pi/4$ did not have a pole with radius within $1._{10}{}^{-3}$ of one to be recognized as a pure sinusoid, and the component is therefore obviously buried in the white noise component.

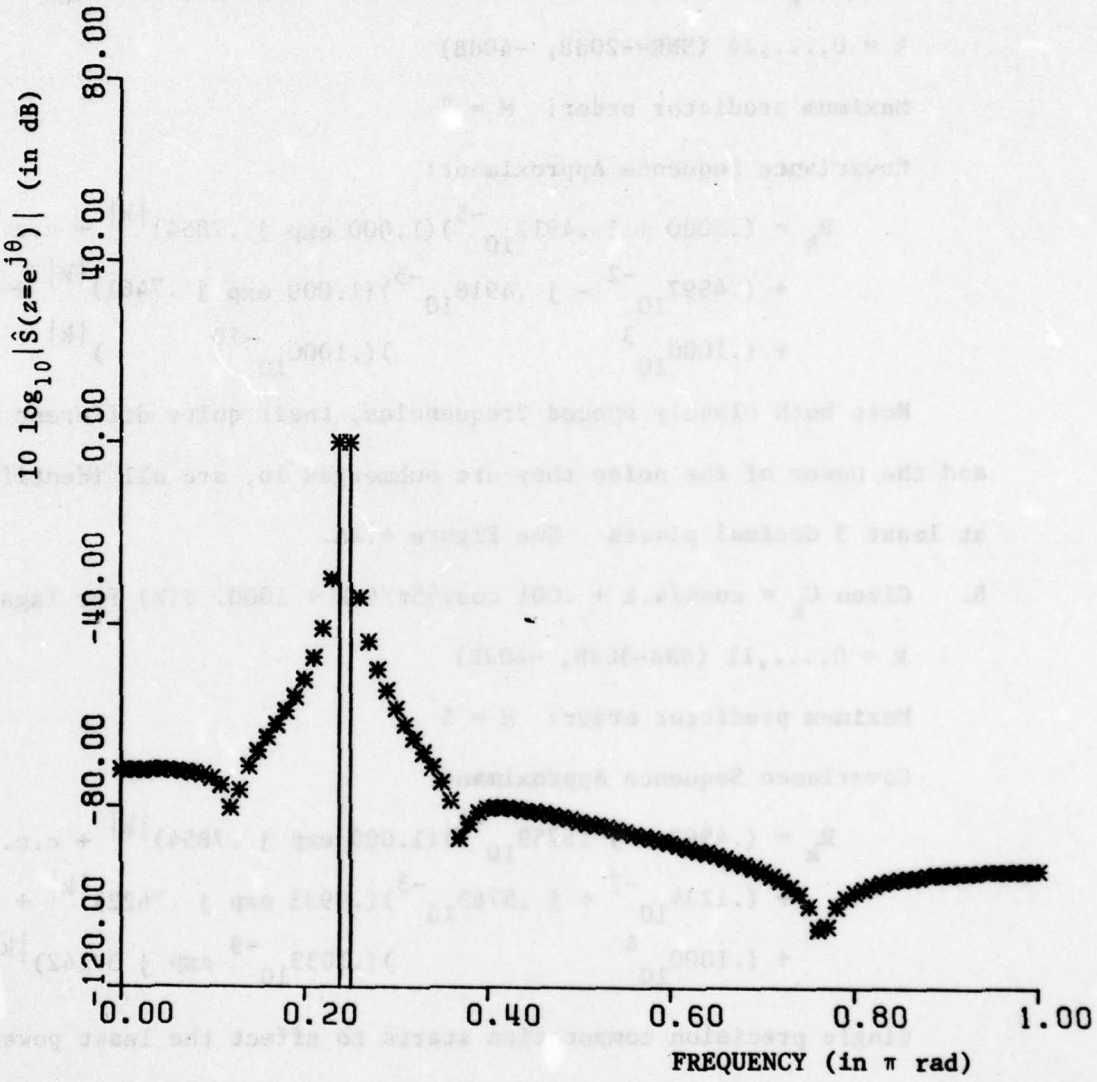

Figure 4.3 Spectral estimate for 2 sines. Given 9 known covariance lags.

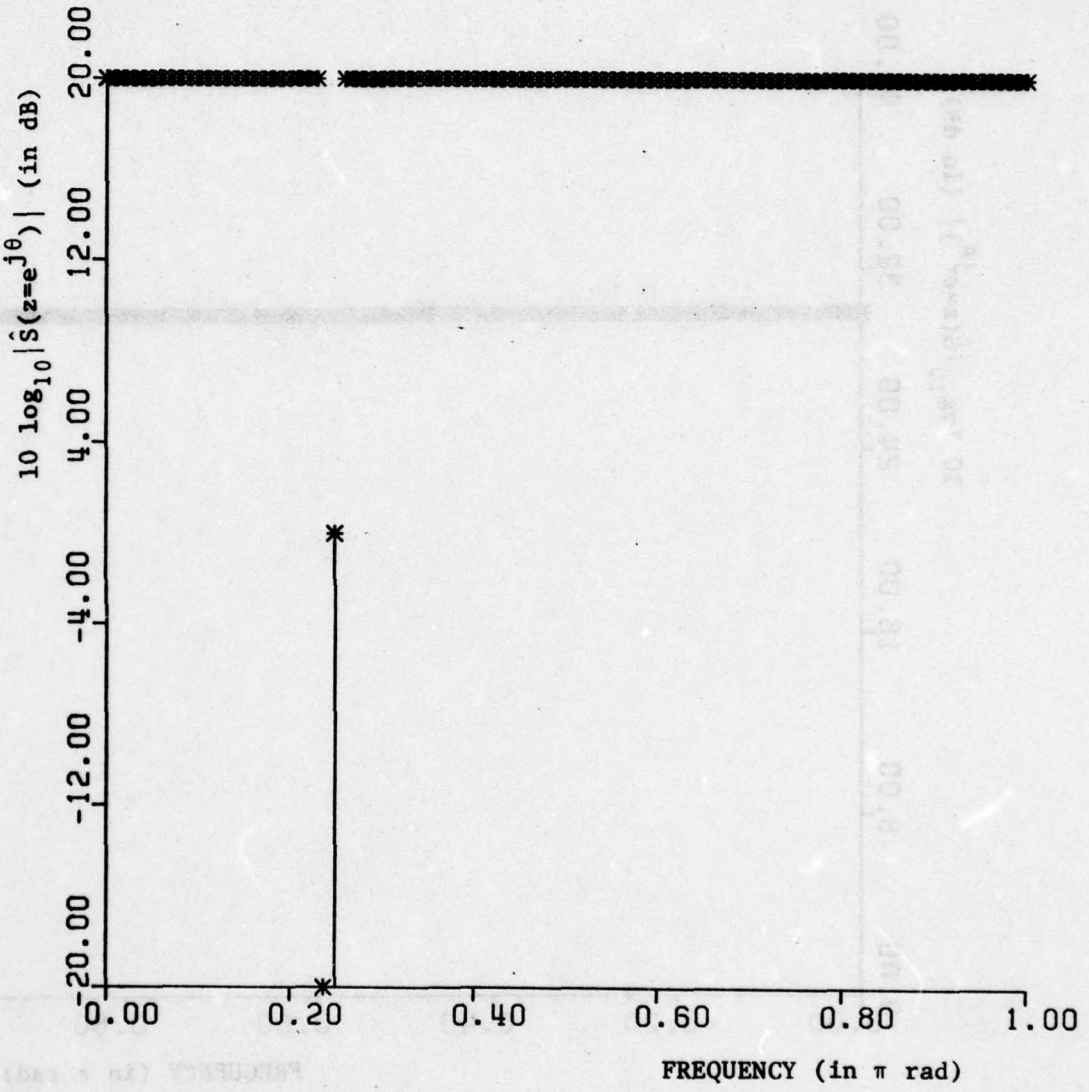

Figure 4.4A Spectral estimate for 2 sines in white noise. Given 15 known covariance lags. SNR = -20 dB, -40 dB.

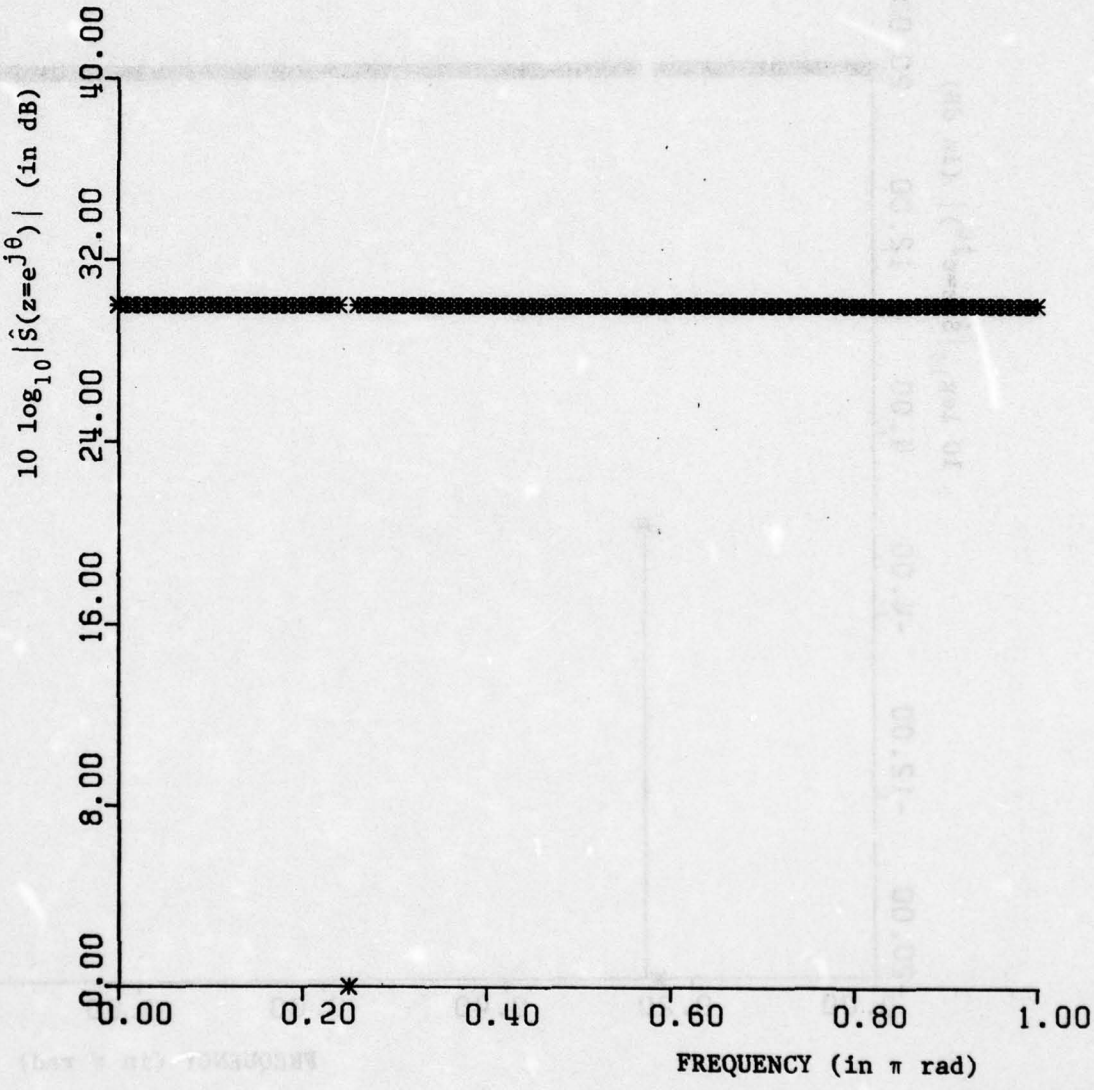

Figure 4.4B Spectral estimate for 2 sines in white noise. Given 12 known covariance lags. SNR = -30 dB, -60 dB.

Example 5 A sinusoid in autoregressive noise

A. Given $C_k = \cos\pi/4.k + C_k^n$ for lags k = 0,...,19

where C_k^n is the covariance sequence for the process resulting from driving

$$H(z) = \frac{1}{1-.8z^{-1}} \qquad 4.23$$

with white noise (0,1). It is easily derived that

$$C_o^n = \frac{1}{1-.8^2} = 2.778$$

Maximum predictor order: M = 5

Covariance Sequence Approximant:

$$R_k = (.5000 + j\ .3921_{10}{}^{-7})(1.000 \exp j\ .7854)^{|k|} + c.c.$$
$$+ (2.778\qquad\qquad)(.8000\qquad\qquad)^{|k|}$$

The frequency estimate, corresponding power and also the first order autoregressive noise are easily seen to be accurate to at least 4 decimal places. The visual spectral information is given in Figure 4.5A. Note that the maximum order was not used here, since the algorithm found accurate representation with order 3.

B. As above but now C_k^n is the covariance sequence resulting from white noise (0,1) input to

$$H(z) = \frac{1}{(1-.8iz^{-1})(1+.8iz^{-1})} \qquad 4.24$$

The sequence C_k^n was generated according to the procedure outlined in [D]. Covariance Sequence Approximant:

$$R_k = (.5000 + j\ .1688_{10}{}^{-8})(1.000 \exp -j\ .7854)^{|k|} + c.c.$$
$$+ (.8469 + j\ .6195_{10}{}^{-8})(.8000 \exp -j\ 1.571)^{|k|} + c.c.$$
$$+ (.1433_{10}{}^{-7}\qquad\qquad)(.7378 \exp -j\ 3.142)^{|k|}$$

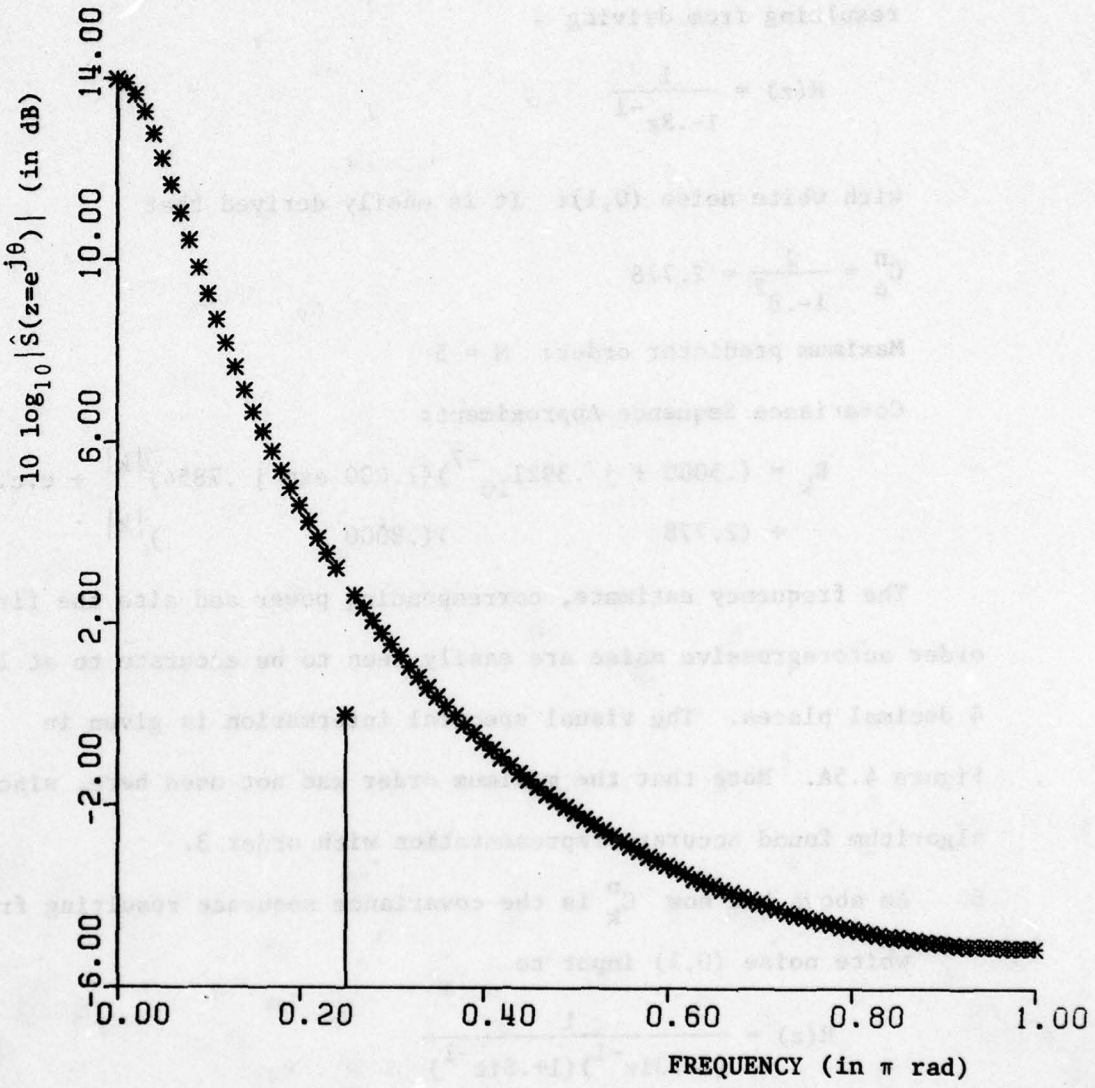

Figure 4.5A Spectral estimate for sine in first order autoregressive noise. Given 20 known covariance lags.

All estimates were found to be accurate to at least 4 decimal places. A spurious pole was identified here, but the correction is achieved by finding an appropriate negligible linear weight factor of order 10^{-8}. The spectral information is depicted in Figure 4.5B.

Example 6 A nonrational spectral density

A. Given $C_k = \exp(-.1k^2)$ for lags $k = 0,\ldots,34$

which represents the Gaussian spectral density. Since it is nonrational, we know there is not any ARMA(N,M) that represents it exactly. This example therefore serves as an experimental test for the robustness of the covariance sequence approximant.

Maximum predictor order: M = 10

Covariance Sequence Approximant:

$$R_k = (.3724_{10}{}^3 + j\ .1771_{10}{}^3)(.1536 \exp j\ .7666)^{|k|} + c.c.$$
$$+ (-.3147_{10}{}^3 - j\ .8752_{10}{}^3)(.1536 \exp j\ .2503)^{|k|} + c.c.$$
$$+ (.5722_{10}{}^2 - j\ .4282_{10}{}^2)(.1533 \exp -j\ 1.348)^{|k|} + c.c.$$

Suppose the sampled covariance sequence $\{C_k\}$ is derived from the continuous covariance function

$$\rho(\tau) = e^{-\alpha^2\tau^2} \qquad 4.25$$

by sampling at $\tau_k = k$, $k = 0,\pm1,\ldots$. The spectral density function associated with $\rho(\tau)$ is

$$S(\omega) = \frac{\sqrt{\pi}}{\alpha}\ e^{-(\omega^2/4\alpha^2)} \qquad 4.26$$

With $\tau_k = k$ the sampling frequency equals one and the spectral density of (4.26) is down 107 dB at the half sampling frequency relative to the zero frequency. The aliasing effect in the infinite sum of translates of the continuous time spectrum is therefore negligible, and we can use

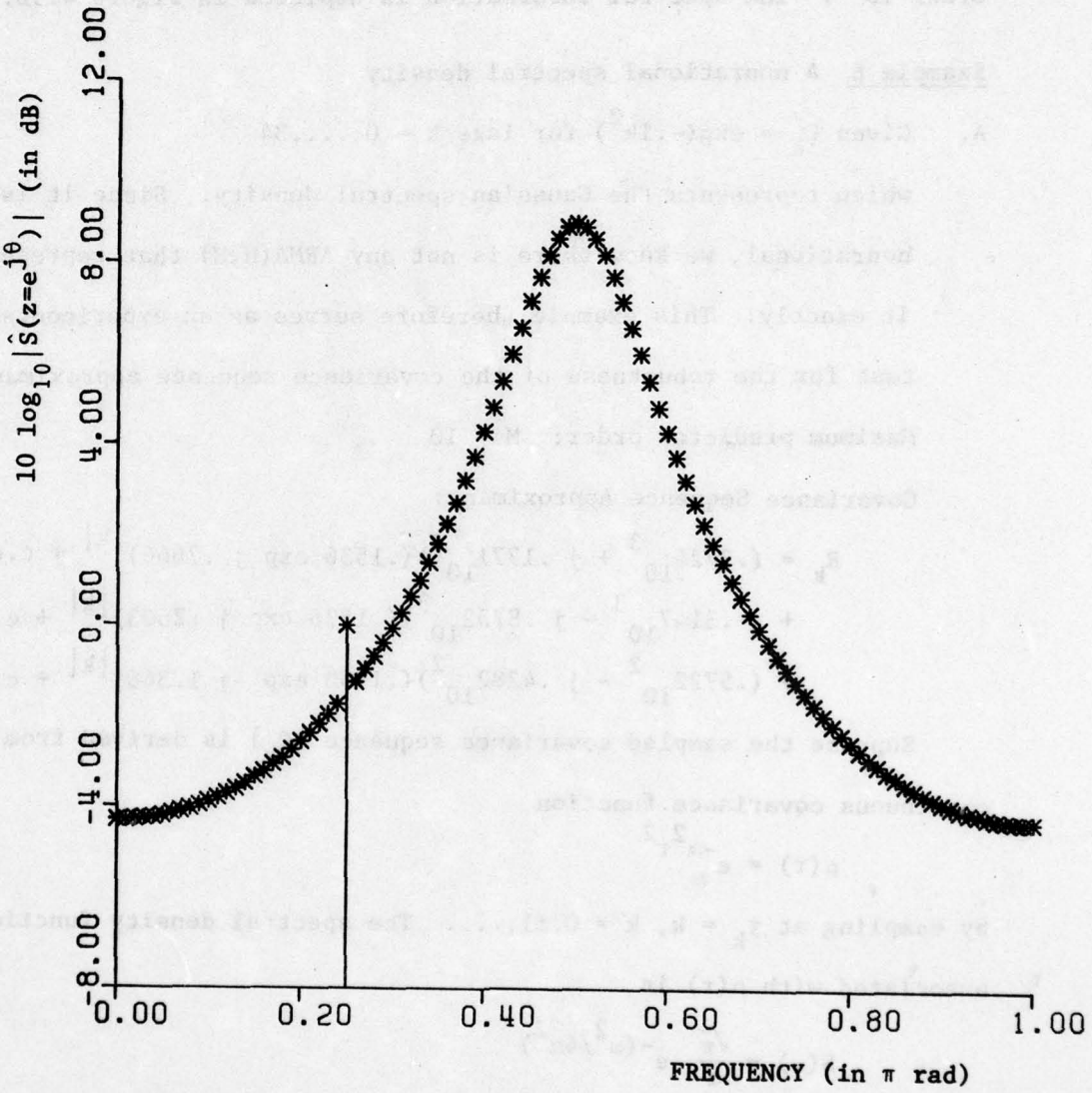

Figure 4.5B Spectral estimate for sine in second order autoregressive noise. Given 20 known covariance lags.

continuous time spectral density values in Table 4.1 to get some idea of the performance of the covariance sequence approximant procedure.

ω(in π rad)	10 $\log_{10} S(\omega)$(in dB)
0.	7.49
.05	7.22
.10	6.41
.15	5.09
.20	3.20
.25	.79
.30	-2.16
.35	-5.64
.40	-9.66
.45	-14.2
.50	-19.3
.60	-31.1
.70	-45.0
.80	-61.1
.90	-79.3
1.0	-99.7

Table 4.1 Gaussian spectral density values

The spectral information in Figure 4.6A shows that the estimate is fairly good up to about $\omega = .5\pi$ rad. Sidelobes then occur at approximately -30dB relative to the peak spectral density. Where the sidelobes occur the spectral density estimate is negative for some frequencies.

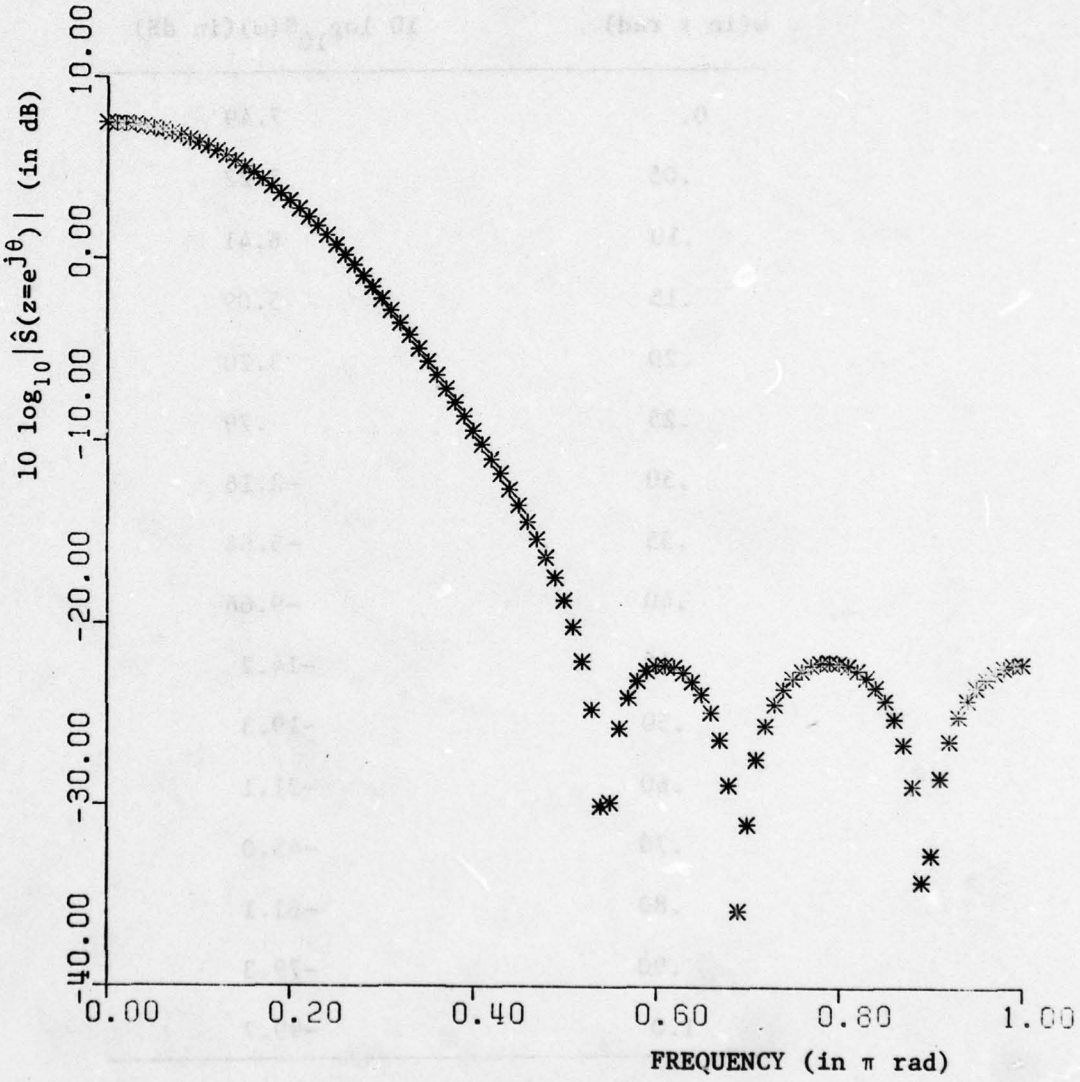

Figure 4.6A Spectral estimate for Gaussian spectral density. Given 35 known covariance lags.

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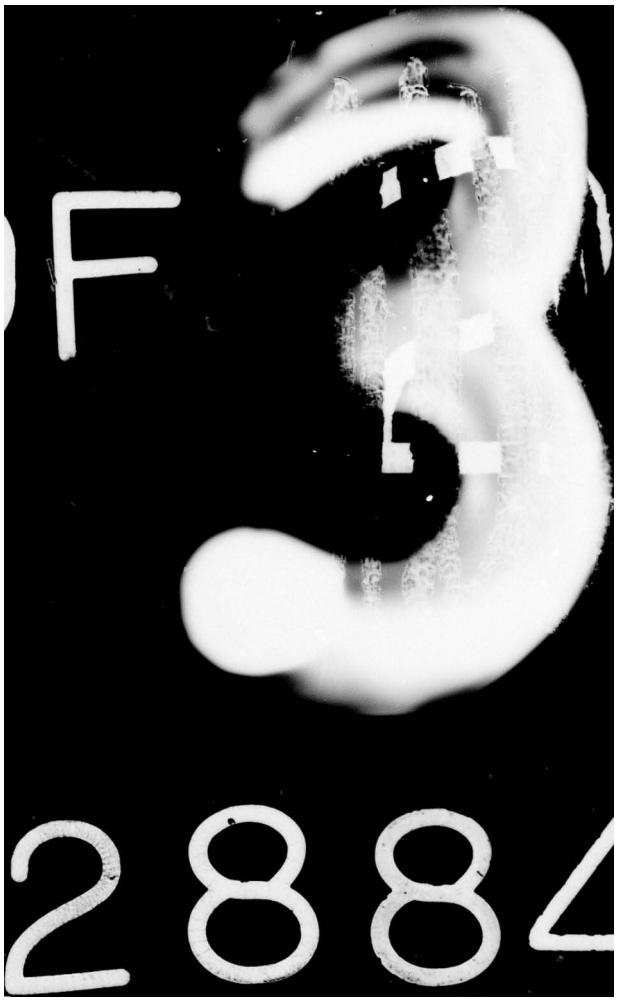
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B. Given $C_k = \exp(-.1k^2)$ for lags $k = 0,\ldots,16$

Maximum predictor order: $M = 7$

Note that the major difference between this example and example 6A is the use of about half as many covariance lags.

Covariance Sequence Approximant:

$$R_k = (4.348 + j\ .6075)(.2644 \exp j\ .9222)^{|k|} + c.c.$$
$$+ (-.2081_{10}{}^{2} - j\ 2.856)(.2634 \exp j\ .4498)^{|k|} + c.c.$$
$$+ (-.2002 + j\ .5371_{10}{}^{-1})(.2662 \exp j\ 1.462)^{|k|} + c.c.$$
$$+ (.3432_{10}{}^{2})(.2631)^{|k|}$$
$$+ (.3884_{10}{}^{-4})(.1000_{10}{}^{-10})^{|k|}$$

All 7 poles were used, resulting in an almost perfect fit of all the given covariance sequence lags. The automatically added noisepole got an appropriate negligible weight of order 10^{-5}. We also note here that no negative spectral density estimate was encountered in this case. Table 4.1 indicates a very accurate spectral density for frequencies up to $\omega = .8\pi$ rad, after which the cumulative aliasing effect becomes noticeable. See Figure 4.6B for the spectral plot.

If perfectly known covariance sequences are available, often only a finite part is necessary to determine an underlying ARMA(M,M^-) model that implicitly extends the observed covariance sequence for all lag values. For most examples considered so far, an ARMA(M,M^-) process or a limiting version of it, was the underlying process. All results were obtained with single precision arithmetic. A nonrational spectral density, such as the Gaussian one, can be approximated very closely with an estimated underlying ARMA(M,M^-). The quite successful rational approximation points to a certain degree of robustness for the covariance sequence approximant and its associated spectral estimate. We also

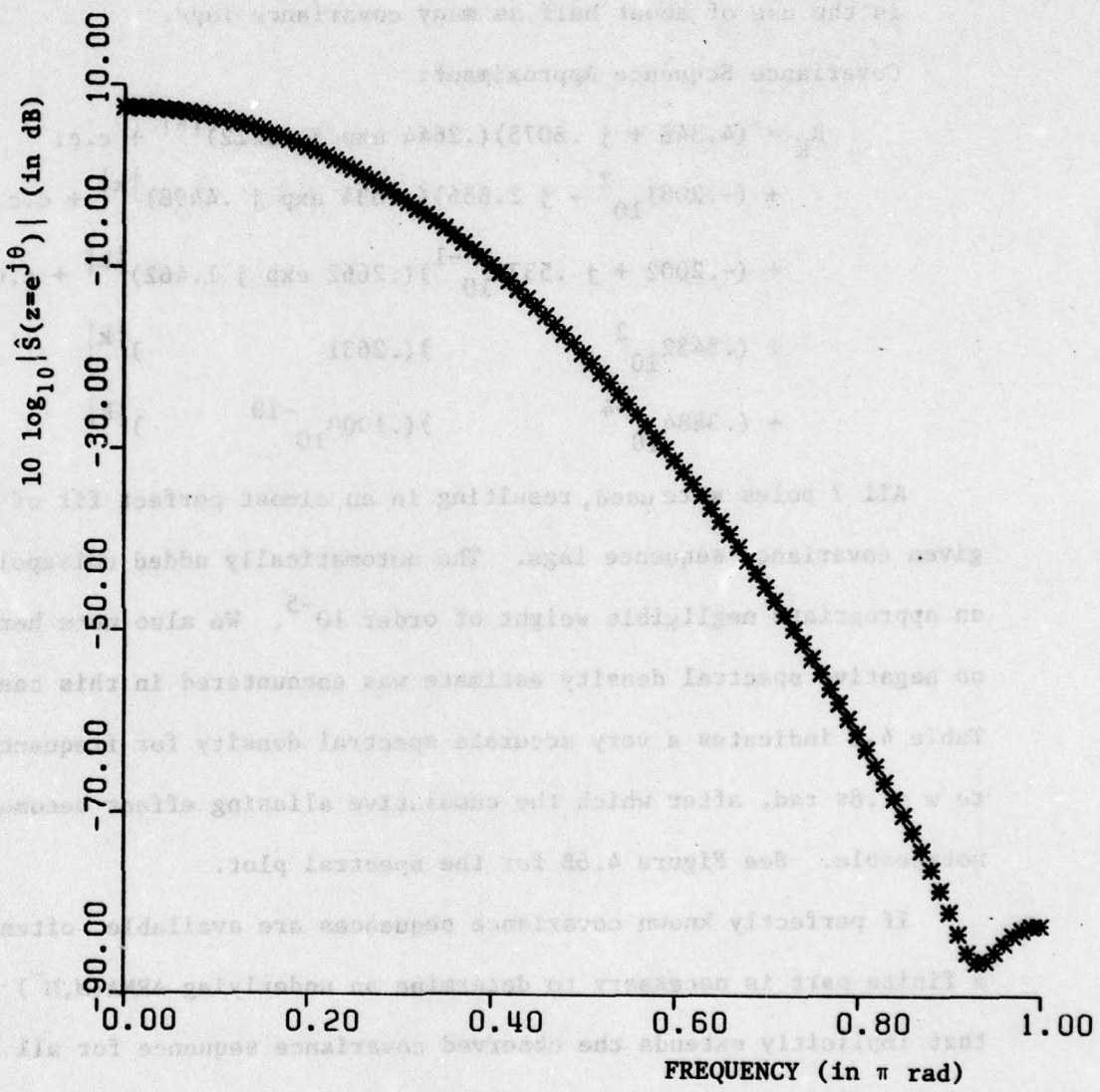

Figure 4.6B Spectral estimate for Gaussian spectral density. Given 17 known covariance lags.

note that in none of the examples for known covariance sequences was an unstable ARMA(M,$\bar{M}$) identified as the approximant.

4.5 Numerical Results for Estimated Covariance Sequences

It is now logical, and very much of practical significance, to evaluate the covariance sequence approximant when operating on a finite length covariance sequence estimate derived from a finite length realization of an observed process. In the examples to follow realizations are generated for several stochastic processes, with and without additive noise. From these realizations a finite length covariance sequence estimate is determined, which will be used to find the covariance sequence approximant and corresponding spectral information.

Unless otherwise indicated, a 100 sample realization is generated from the following algorithm:

$$X_i = \sin(\pi/4.i) + S2\ \cos(\text{frac}\ \pi/4.i) + SN\ n_i \qquad 4.27$$

where S2 represents the amplitude of a second frequency component, which has a frequency either $.95\pi/4$ or $.6\pi/4$, and where SN controls the power of a (0,1) white Gaussian noise generator. The power at the frequency $\pi/4$ is kept at 1/2, and the power at the second frequency is given by $1/2 \cdot (S2)^2$, whereas the noisepower is given by $(SN)^2$. From the finite length realization generated by (4.27) a covariance sequence estimate is determined according to the following time averaging algorithm:

$$\hat{C}_k = \frac{1}{100-|k|} \sum_{i=1}^{100-|k|} X_i X_{i+k};\ k = 0,\ldots,19 \qquad 4.28$$

The estimator was chosen for its property of unbiasedness. The number of lags is a purely heuristic decision. If in the estimator one divides by the total number of data points, a biased covariance sequence

estimator is obtained, and the covariance sequence approximant for it yields extra damping of the pole locations, as is to be expected from the windowing effect. The maximum order in the following examples was arbitrarily set at seven. In this way we adhere to the philosophy that one should avoid fitting the stochastic covariance sequence estimate exactly. With maximum order seven and the use of twenty covariance sequence lag values the fit is least squares.

The covariance sequence approximant will form an approximation to the covariance sequence estimate, and it therefore seems logical to investigate the behavior of the covariance sequence estimate derived from a finite data record. Let us first determine that behavior for a finite observation of a pure sinusoid:

$$X_i = A \sin(\theta k+\phi); \qquad \theta \neq 0 \tag{4.29}$$

Defining the operator ε_i^N with the following relation

$$\varepsilon_i^N\{X_k X_{k+i}\} = \frac{1}{N - |i|} \sum_{k=1}^{N-|i|} X_k X_{k+i} \tag{4.30}$$

and using trigonometric identities, we derive for the covariance sequence estimate

$$\begin{aligned}
\hat{C}_i &= A^2 \varepsilon_i^N\{X_k X_{k+i}\} \\
&= A^2 \varepsilon_i^N\{\sin\theta k+\phi \ \sin(k+i)\theta+\phi\} \\
&= \frac{A^2}{2} \varepsilon_i^N\{\cos\theta i - \cos(2k\theta+\theta i+2\phi)\} \\
&= \frac{A^2}{2} \cos\theta i - \frac{A^2}{2} \varepsilon_i^N\{\cos 2k\theta+2\phi \cdot \cos\theta i - \sin 2k\theta+2\phi \cdot \sin\theta i\} \\
&= \frac{A^2}{2} \cos\theta i \left[1 - \varepsilon_i^N\{\cos 2k\theta+2\phi\}\right] + \frac{A^2}{2} \sin\theta i \varepsilon_i^N\{\sin 2k\theta+2\phi\}
\end{aligned} \tag{4.31}$$

The covariance sequence is seen to consist of two orthogonal components. The cosθi component converges to the theoretically expected covariance sequence, and the sinθi component converges simultaneously to zero. For 100 datapoints we can bound the error terms by 1 percent of the summed value of samples over half a period of the double frequency 2θ. As the covariance sequence approximant has the versatility to represent the sinθi component, we should expect to see a good approximation to the expression in (4.31), rather than a good approximation to the theoretical covariance sequence for the process in (4.29) based on ensemble averaging and given by:

$$C_i = \frac{1}{2} A^2 \cos\theta i \qquad 4.32$$

Note that the model underlying the Pisarenko decomposition cannot represent the orthogonal sinθi component associated with the covariance sequence estimate of (4.31). The magnitude of the orthogonal error component furthermore can give an indication of the error term affecting the theoretically expected cosθi term. If the summation operator on sin2kθ samples yields zero because it operates over an integer number of periods, then the same summation operator on cos2kθ samples also yields zero. In the latter case the magnitude of the cosθi component gives an indication of the power in the sinusoid.

Example 7 A single sinusoid

S2 = SN = 0

Covariance Sequence Approximant:

$$R_k = (.8972_{10}{}^{-5} - j\ .4392_{10}{}^{-3})(1.034 \exp j\ .7851)^{|k|} + \text{c.c.}$$

$$+ (.2500 + j\ .2061_{10}{}^{-2})(1.000 \exp -j\ .7854)^{|k|} + \text{c.c.}$$

$$+ (-.5995_{10}{}^{-9} + j\ .3025_{10}{}^{-9})(.7555 \exp j\ 2.608)^{|k|} + \text{c.c.}$$

$$+ (.4079_{10}{}^{-8})(.3151 \exp j\ 3.142)^{|k|}$$

$$+ (-.3161_{10}{}^{-7})(.1000_{10}{}^{-10})^{|k|}$$

As indicated in (4.31) an orthogonal component could be expected with magnitude less than 1 percent of the true power, which is seen to be valid. The frequency estimate is seen to be accurate to 4 decimal places. Several spurious poles were identified but their respective linear weights are very small. Note the occurrence of an unstable pole close to the dominant frequency, which is then discounted by a very small linear weight coefficient. The spectral information depicted in Figure 4.7 was computed according to the algorithmic expression for the parametric spectrum estimate in (4.2). A spectrum does not exist in this case because $\{R_k\}$ is explosive by virtue of the poles with radius 1.034. For an unstable pole pair however that computation yields the negative of the spectral density for the stable pole pair associated with the reflection of the unstable poles. For this particular covariance sequence approximant the unstable pole pair forms the only major contribution to a "continuous spectral density" which results in the deceptively regular appearance of Figure 4.7.

The next question to be answered is what happens when more sinusoidal components are present? To provide at least a partial

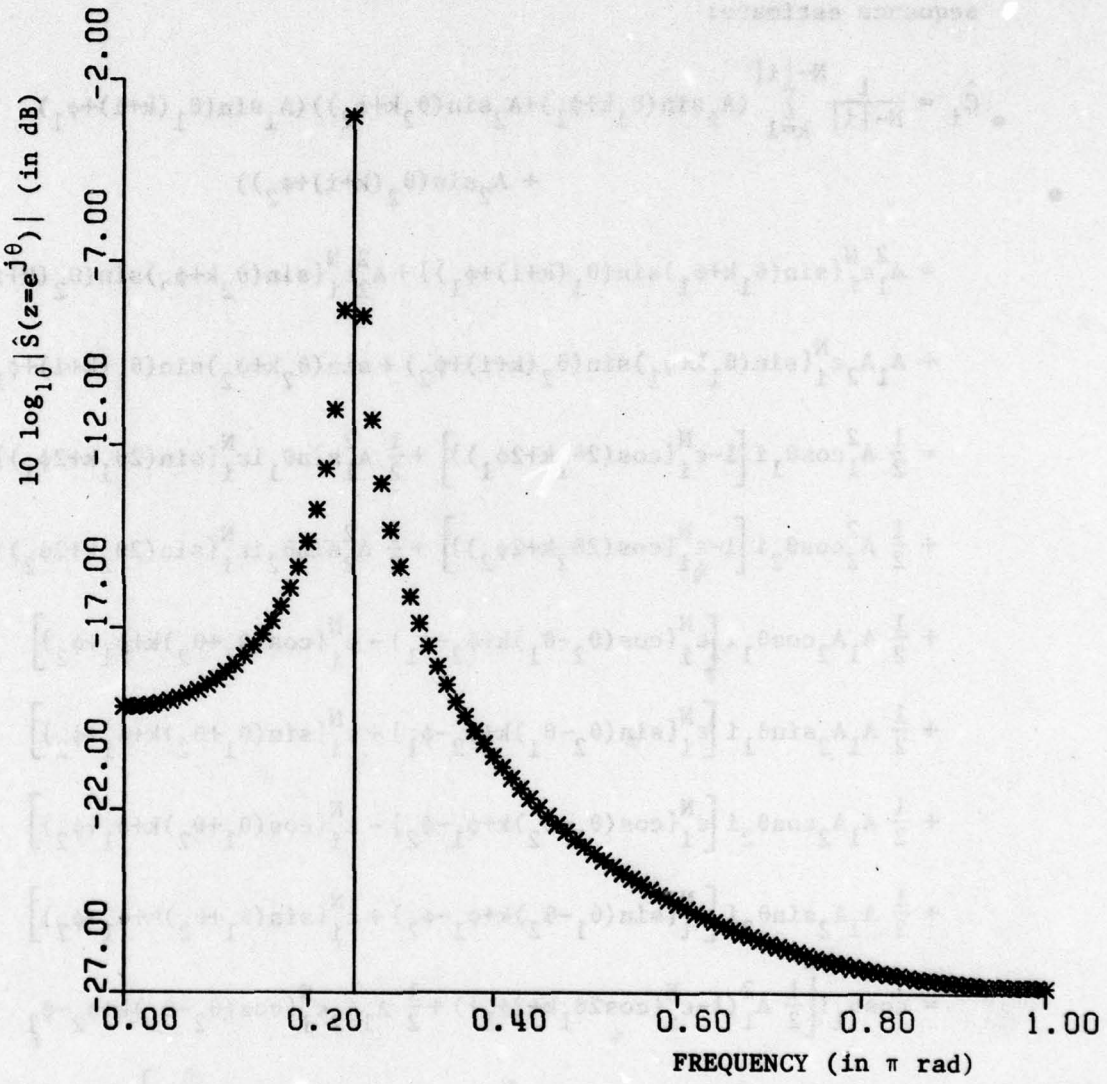

Figure 4.7 Spectral estimate for sine. Estimated 20 covariance lags from 100 samples.

answer we examine the following realization:

$$X_i = A_1\sin(\theta_1 i+\phi_1) + A_2\sin(\theta_2 i+\phi_2) \qquad 4.33$$

Repeated use of trigonometric identities yields the following covariance sequence estimate:

$$\hat{C}_i = \frac{1}{N-|i|}\sum_{k=1}^{N-|i|}(A_1\sin(\theta_1 k+\phi_1)+A_2\sin(\theta_2 k+\phi_2))(A_1\sin(\theta_1(k+i)+\phi_1) + A_2\sin(\theta_2(k+i)+\phi_2))$$

$$= A_1^2\varepsilon_i^N\{\sin(\theta_1 k+\phi_1)\sin(\theta_1(k+i)+\phi_1)\} + A_2^2\varepsilon_i^N\{\sin(\theta_2 k+\phi_2)\sin(\theta_2(k+i)+\phi_2)\}$$

$$+ A_1A_2\varepsilon_i^N\{\sin(\theta_1 k+\phi_1)\sin(\theta_2(k+i)+\phi_2) + \sin(\theta_2 k+\phi_2)\sin(\theta_1(k+i)+\phi_1)\}$$

$$= \frac{1}{2}A_1^2\cos\theta_1 i\left[1-\varepsilon_i^N\{\cos(2\theta_1 k+2\phi_1)\}\right] + \frac{1}{2}A_1^2\sin\theta_1 i\varepsilon_i^N\{\sin(2\theta_1 k+2\phi_1)\}$$

$$+ \frac{1}{2}A_2^2\cos\theta_2 i\left[1-\varepsilon_i^N\{\cos(2\theta_2 k+2\phi_2)\}\right] + \frac{1}{2}A_2^2\sin\theta_2 i\varepsilon_i^N\{\sin(2\theta_2 k+2\phi_2)\}$$

$$+ \frac{1}{2}A_1A_2\cos\theta_1 i\left[\varepsilon_i^N\{\cos(\theta_2-\theta_1)k+\phi_2-\phi_1\} - \varepsilon_i^N\{\cos(\theta_1+\theta_2)k+\phi_1+\phi_2\}\right]$$

$$+ \frac{1}{2}A_1A_2\sin\theta_1 i\left[\varepsilon_i^N\{\sin(\theta_2-\theta_1)k+\phi_2-\phi_1\} + \varepsilon_i^N\{\sin(\theta_1+\theta_2)k+\phi_1+\phi_2\}\right]$$

$$+ \frac{1}{2}A_1A_2\cos\theta_2 i\left[\varepsilon_i^N\{\cos(\theta_1-\theta_2)k+\phi_1-\phi_2\} - \varepsilon_i^N\{\cos(\theta_1+\theta_2)k+\phi_1+\phi_2\}\right]$$

$$+ \frac{1}{2}A_1A_2\sin\theta_2 i\left[\varepsilon_i^N\{\sin(\theta_1-\theta_2)k+\phi_1-\phi_2\} + \varepsilon_i^N\{\sin(\theta_1+\theta_2)k+\phi_1+\phi_2\}\right]$$

$$= \cos\theta_1 i\left[\frac{1}{2}A_1^2(1-\varepsilon_i^N\{\cos 2\theta_1 k+2\phi_1\}) + \frac{1}{2}A_1A_2\varepsilon_i^N\{\cos(\theta_2-\theta_1)k+\phi_2-\phi_1 - \cos(\theta_1+\theta_2)k+\phi_1+\phi_2\}\right]$$

$$+ \sin\theta_1 i\left[\frac{1}{2}A_1^2\varepsilon_i^N\{\sin 2\theta_1 k+2\phi_1\} + \frac{1}{2}A_1A_2\varepsilon_i^N\{\sin(\theta_2-\theta_1)k+\phi_2-\phi_1 + \sin(\theta_1+\theta_2)k+\phi_1+\phi_2\}\right]$$

$$+ \cos\theta_2 i\left[\frac{1}{2}A_2^2(1-\varepsilon_i^N\{\cos 2\theta_2 k+2\phi_2\}) + \frac{1}{2}A_1A_2\varepsilon_i^N\{\cos(\theta_1-\theta_2)k+\phi_1-\phi_2 - \cos(\theta_1+\theta_2)k+\phi_1+\phi_2\}\right]$$

$$+ \sin\theta_2 i\left[\frac{1}{2} A_2^2 \varepsilon_i^N\{\sin 2\theta_2 k + 2\phi_2\} + \frac{1}{2} A_1 A_2 \varepsilon_i^N\{\sin(\theta_1-\theta_2)k + \phi_1 - \phi_2 + \sin(\theta_1+\theta_2)k + \phi_1 + \phi_2\}\right] \quad 4.34$$

Note again that a decomposition into orthogonal components takes place and furthermore that all error terms involving $\varepsilon_i^N\{\cdot\}$ converge to zero, if $\theta_1 \neq \theta_2 \neq 0$. Thus the covariance sequence estimate based on a single realization is an asymptotically unbiased estimator for the covariance sequence.

For the examples to follow a process realization was generated with the algorithm of (4.27) so that in (4.33):

$$\begin{aligned} A_1 &= 1 \quad , A_2 = S_2 \\ \theta_1 &= \pi/4, \ \theta_2 = \text{frac } \pi/4 \\ \phi_1 &= 0 \quad , \phi_2 = \pi/2 \end{aligned} \quad 4.35$$

Example 8 Two sinusoids

A. $S_2 = 1$

frac = .95

SN = 0

Covariance Sequence Approximant:

$$R_k = (.3611 - j\ .1508)(.9989 \exp j\ .7819)^{|k|} + c.c.$$

$$+ (.3581 - j\ .1535)(.9988 \exp -j\ .7497)^{|k|} + c.c.$$

$$+ (.4111_{10}{}^{-5}\qquad)(.8340\qquad)^{|k|}$$

$$+ (.1321_{10}{}^{-5} - j\ .3982_{10}{}^{-6})(.5592 \exp j\ 2.443)^{|k|} + c.c.$$

$$+ (-.5151_{10}{}^{-5}\qquad)(.1000_{10}{}^{-10}\qquad)^{|k|}$$

Both frequency estimates are correct to within 1/2 percent. One might expect that this leads to a very good overall estimate, and indeed all twenty lags are matched to at least 4 decimal places. Spurious poles have indeed an appropriate low linear weighting coefficient. The power estimates however are substantially too high. From (4.34) we see that a double frequency and sum frequency component enters as error term, and by the same argument as before they are of the order of 1 percent. The more important error term in this case could be the difference frequency term. This difference frequency is $\pi/4 - .95\pi/4 = \pi/80$. With a 100 point data sequence this turns out to be an almost worst case situation where the error terms are very large. The covariance sequence approximant above merely satisfies the covariance sequence estimate. The spectral information is depicted in Figure 4.8A. Note that there is no spectral resolution readily available in the plot, whereas the covariance sequence approximant clearly indicates two strong sinusoidal components.

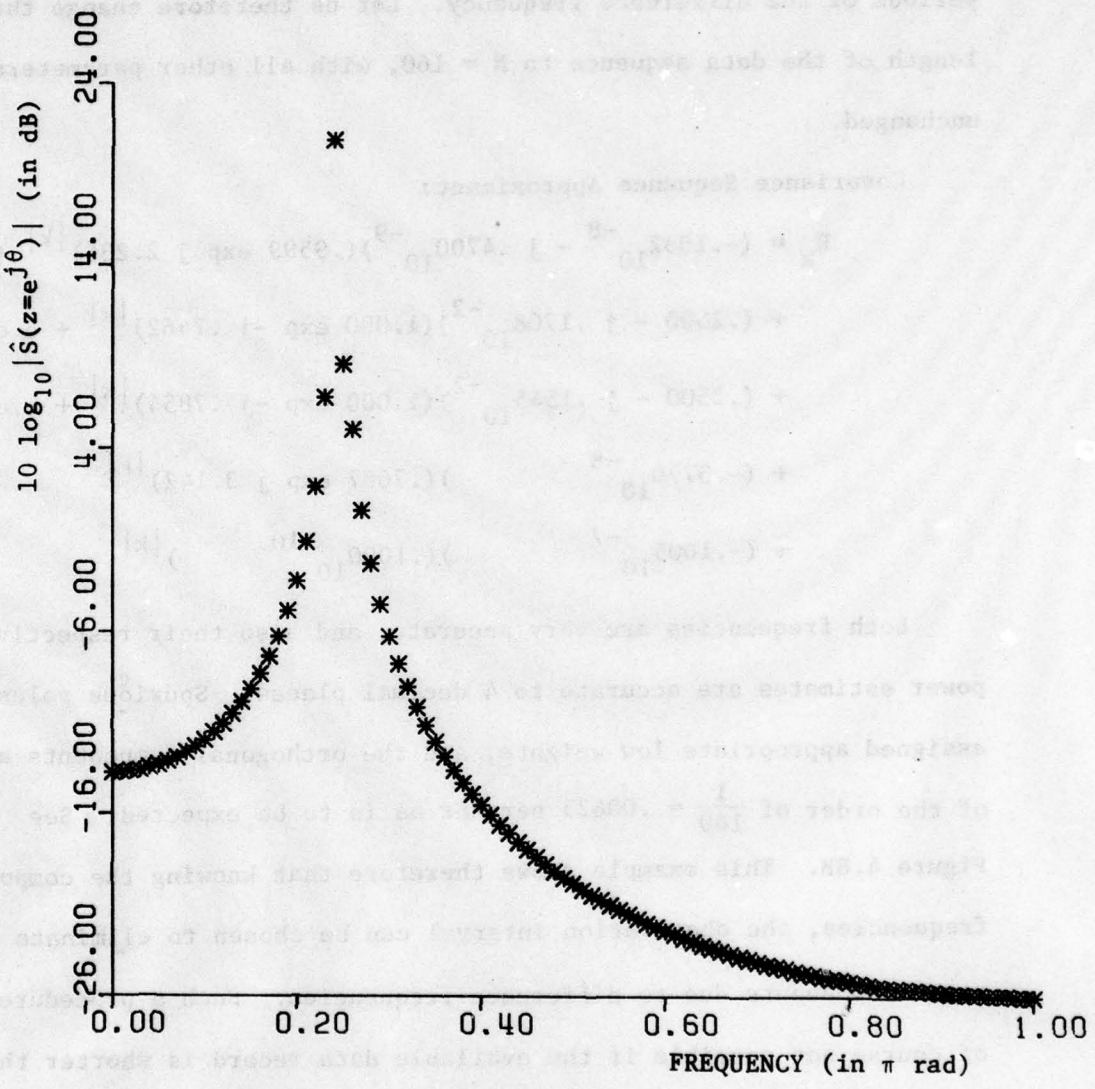

Figure 4.8A Spectral estimate for 2 sines. Estimated 20 covariance lags from 100 samples.

B. Based on the frequency estimates from example 8A and the error terms indicated in (4.34), we can decide to minimize these error terms by insuring that summation takes place over an integer number of periods of the difference frequency. Let us therefore change the length of the data sequence to N = 160, with all other parameters unchanged.

Covariance Sequence Approximant:

$$R_k = (-.1032_{10}{}^{-8} - j\ .4700_{10}{}^{-9})(.9599 \exp j\ 2.236)^{|k|} + c.c.$$
$$+ (.2500 - j\ .1708_{10}{}^{-2})(1.000 \exp -j\ .7462)^{|k|} + c.c.$$
$$+ (.2500 - j\ .1545_{10}{}^{-2})(1.000 \exp -j\ .7854)^{|k|} + c.c.$$
$$+ (-.3770_{10}{}^{-8}\qquad)(.7687 \exp j\ 3.142)^{|k|}$$
$$+ (-.1005_{10}{}^{-7}\qquad)(.1000_{10}{}^{-10}\qquad)^{|k|}$$

Both frequencies are very accurate, and also their respective power estimates are accurate to 4 decimal places. Spurious poles were assigned appropriate low weights, and the orthogonal components are of the order of $\frac{1}{160} = .00625$ percent as is to be expected. See Figure 4.8B. This example shows therefore that knowing the component frequencies, the observation interval can be chosen to eliminate the error components due to difference frequencies. Such a procedure is of course not possible if the available data record is shorter than one period of the difference frequency of interest.

Let us now assume the ultimate practical case where our observation interval is finite and the observations are affected by additive noise. For white noise we then make an estimate of the covariance sequence for lags different from zero:

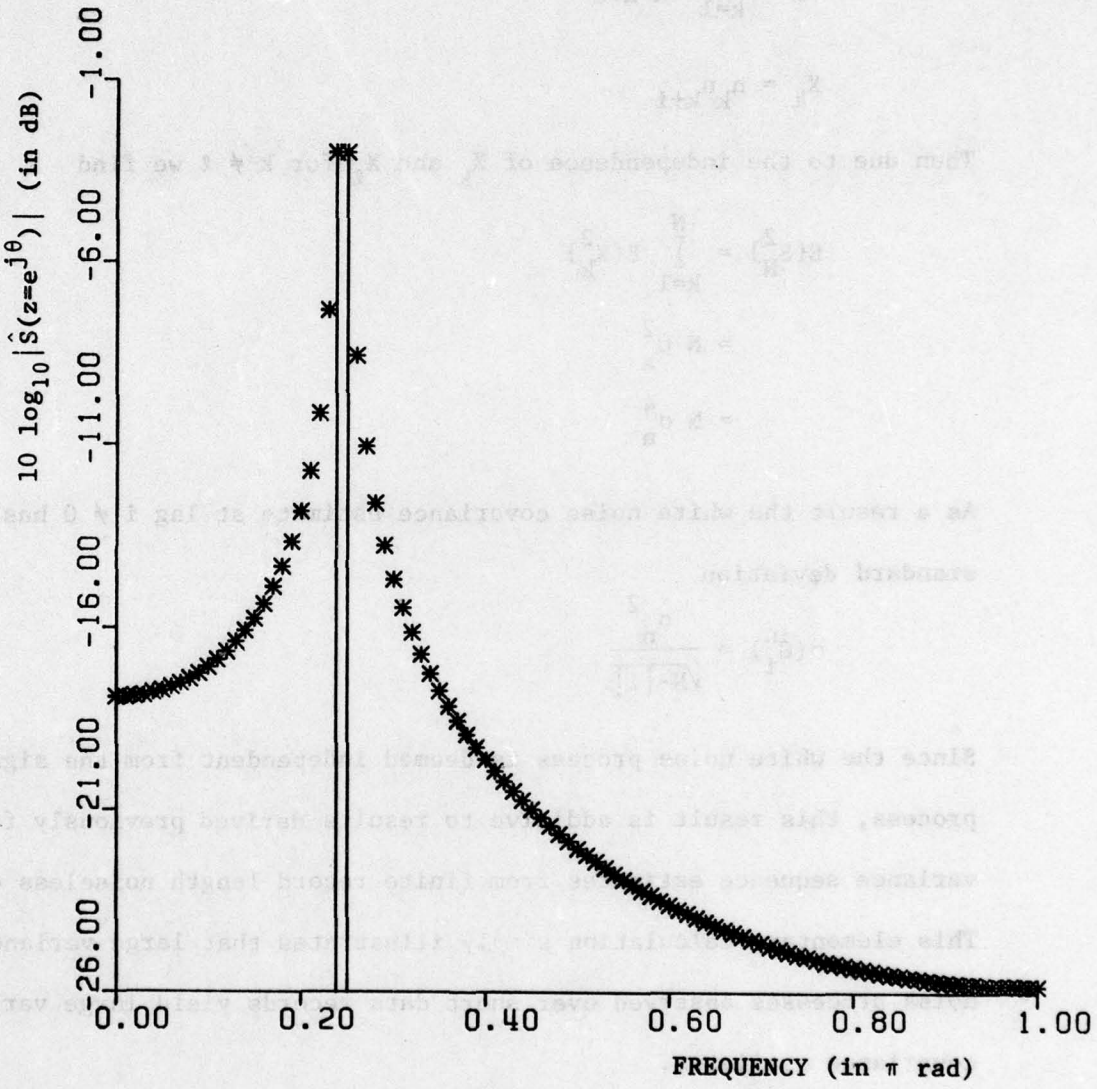

Figure 4.8B Spectral estimate for 2 sines. Estimated 20 covariance lags from 160 samples.

$$\hat{C}_i^n = \varepsilon_i^N\{n_k n_{k+i}\} \tag{4.36}$$

where n_k is white noise, so that $n_k \perp n_{k+i}$. If we define

$$S_N = \sum_{k=1}^{N} n_k n_{k+i} \tag{4.37a}$$

$$X_k = n_k n_{k+i} \tag{4.37b}$$

Then due to the independence of X_k and X_ℓ for $k \neq \ell$ we find

$$\begin{aligned} E\{S_N^2\} &= \sum_{k=1}^{N} E\{X_k^2\} \\ &= N\,\sigma_x^2 \\ &= N\,\sigma_n^4 \end{aligned} \tag{4.38}$$

As a result the white noise covariance estimate at lag $i \neq 0$ has a standard deviation

$$\sigma(\hat{C}_i^n) = \frac{\sigma_n^2}{\sqrt{N-|i|}} \tag{4.39}$$

Since the white noise process is deemed independent from the signal process, this result is additive to results derived previously for covariance sequence estimates from finite record length noiseless data. This elementary calculation simply illustrates that large variance white noise processes observed over short data records yield large variance covariance estimates.

Example 9 A single sinusoid in white noise

A. $S_2 = 0$

SN = .03 (SNR = 27dB)

Covariance Sequence Approximant:

$$R_k = (.2481 - j\ .2472_{10}{}^{-2})(1.000 \exp j\ .7850)^{|k|} + c.c.$$
$$+ (.7103_{10}{}^{-3}\qquad)(.7826\qquad)^{|k|}$$
$$+ (.7122_{10}{}^{-3} + j\ .2098_{10}{}^{-4})(.6216 \exp -j\ 2.668)^{|k|} + c.c.$$
$$+ (.4132_{10}{}^{-2} - j\ .7663_{10}{}^{-3})(.3997 \exp -j\ .9054)^{|k|} + c.c.$$
$$+ (-.7795_{10}{}^{-2}\qquad)(.1000_{10}{}^{-10}\qquad)^{|k|}$$

The sinusoidal pole and its corresponding power are identified quite nicely. Spurious poles are identified and seem to be approximating the noise process. See Figure 4.9A for visual spectral information.

B. $S_2 = 0$

SN = .3 (SNR = 7dB)

Covariance Sequence Approximant:

$$R_k = (.2364 - j\ .2470_{10}{}^{-2})(1.000 \exp j\ .7827)^{|k|} + c.c.$$
$$+ (.2026_{10}{}^{-1}\qquad)(.9295\qquad)^{|k|}$$
$$+ (-.1285_{10}{}^{-1} - j\ .8271_{10}{}^{-2})(.7551 \exp -j\ 2.005)^{|k|} + c.c.$$
$$+ (.1335 + j\ .1051)(.3341 \exp -j\ 2.206)^{|k|} + c.c.$$
$$+ (-.1614\qquad)(.1000_{10}{}^{-10}\qquad)^{|k|}$$

Note the frequency estimate is very accurate and the corresponding power estimate is reasonable. Spurious poles are such that they represent an average noise level of -10dB as well as can be expected. See Figure 4.9B.

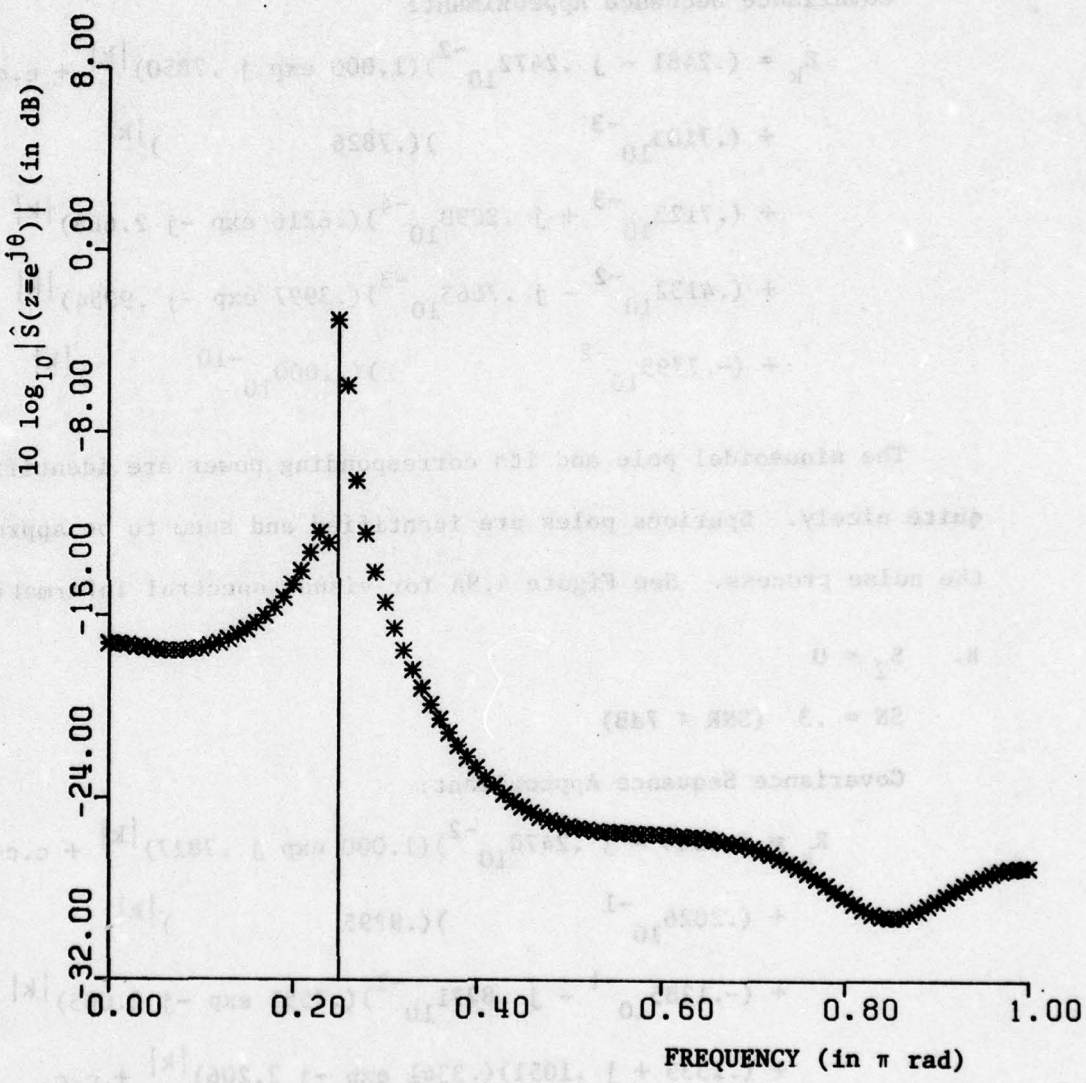

Figure 4.9A Spectral estimate for sine in white noise. SNR = 27 dB. Estimated 20 covariance lags from 100 samples.

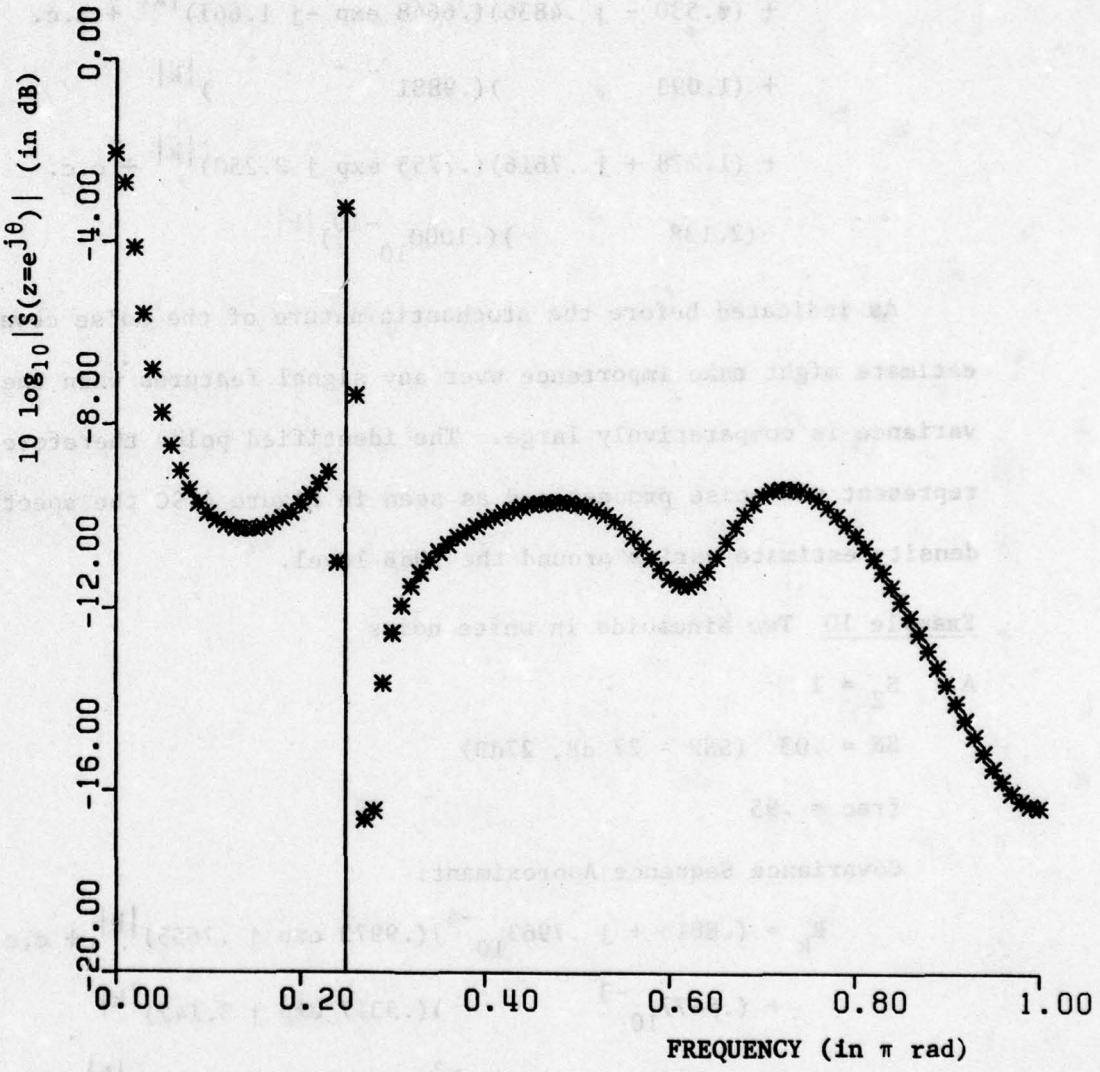

Figure 4.9B Spectral estimate for sine in white noise. SNR = 7 dB. Estimated 20 covariance lags from 100 samples.

C. $S_2 = 0$

SN = .3 (SNR ≈ -13dB)

Covariance Sequence Approximant:

$$R_k = (.5260 - j\ 2.549)(.7368 \exp j\ 1.255)^{|k|} + c.c.$$

$$+ (3.530 - j\ .4836)(.6648 \exp -j\ 1.661)^{|k|} + c.c.$$

$$+ (1.093\qquad\qquad)(.9881\qquad\qquad)^{|k|}$$

$$+ (1.278 + j\ .7616)(.7755 \exp j\ 2.250)^{|k|} + c.c.$$

$$-(2.138\qquad\qquad)(.1000_{10}{}^{-10})^{|k|}$$

As indicated before the stochastic nature of the noise covariance estimate might take importance over any signal features when the noise variance is comparatively large. The identified poles therefore represent the noise process and as seen in Figure 4.9C the spectral density estimate varies around the 10dB level.

Example 10 Two sinusoids in white noise

A. $S_2 = 1$

SN = .03 (SNR ≈ 27 dB, 27dB)

frac = .95

Covariance Sequence Approximant:

$$R_k = (.8815 + j\ .7963_{10}{}^{-4})(.9973 \exp j\ .7655)^{|k|} + c.c.$$

$$+ (.5477_{10}{}^{-3}\qquad\qquad)(.5317 \exp j\ 3.142)^{|k|}$$

$$+ (-.1608 + j\ .2233_{10}{}^{-2})(.9605 \exp j\ .7630)^{|k|} + c.c.$$

$$+ (-.5694_{10}{}^{-5} + j\ .2818_{10}{}^{-3})(.6813 \exp -j\ 2.226)^{|k|} + c.c.$$

$$+ (.7627_{10}{}^{-3}\qquad\qquad\qquad)(.1000_{10}{}^{-10}\qquad)^{|k|}$$

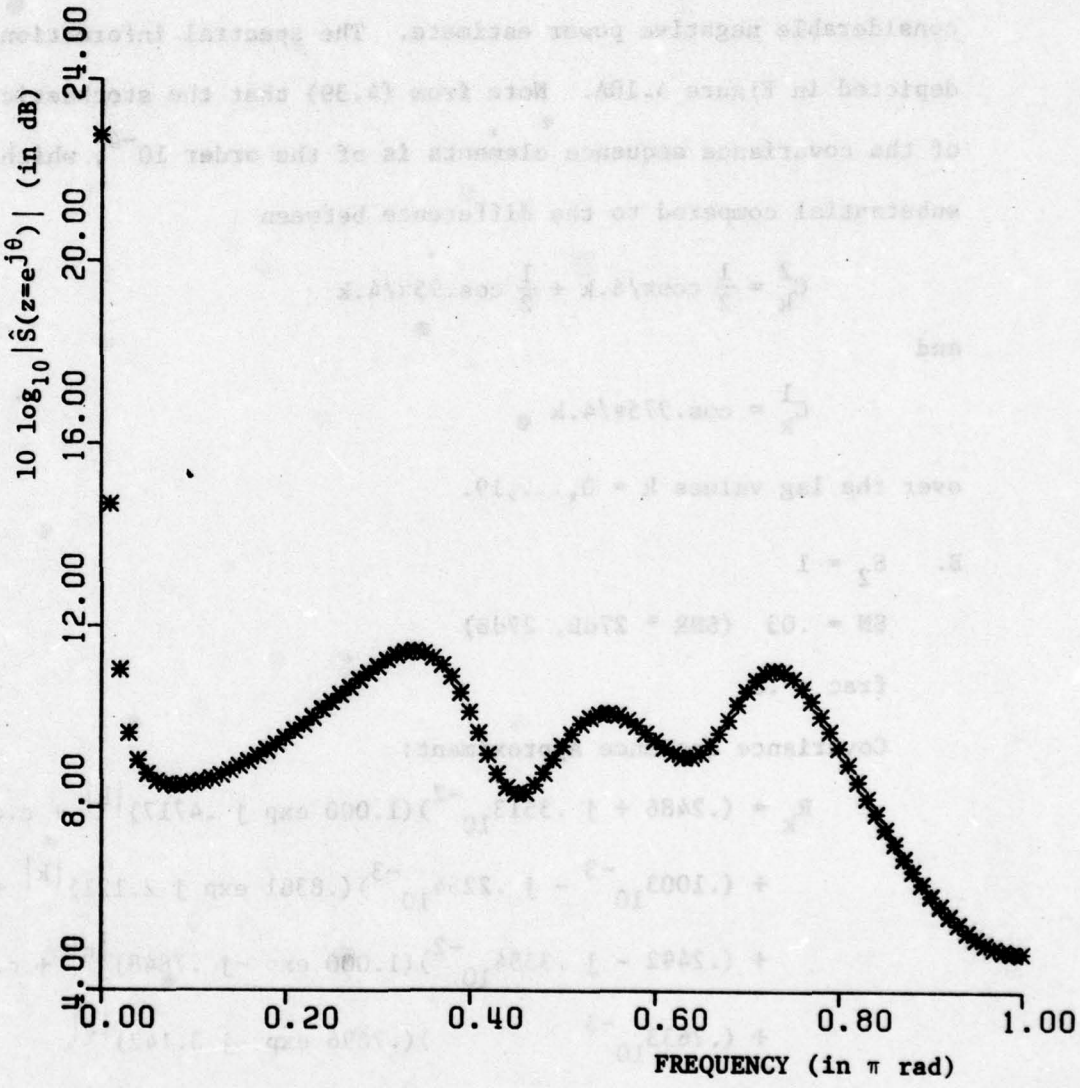

Figure 4.9C Spectral estimate for sine in white noise. SNR = -13 dB. Estimated 20 covariance lags from 100 samples.

No resolution of the frequency components is obtained. Instead the frequencies are virtually averaged, which may then cause a large deviation in the respective power estimates due to error terms dependent on the difference frequency. In this case one of the frequencies shows a considerable negative power estimate. The spectral information is depicted in Figure 4.10A. Note from (4.39) that the stochastic element of the covariance sequence elements is of the order 10^{-4}, which is substantial compared to the difference between

$$C_k^2 = \frac{1}{2}\cos\pi/4.k + \frac{1}{2}\cos.95\pi/4.k$$

and

$$C_k^1 = \cos.975\pi/4.k$$

over the lag values k = 0,...,19.

B. $S_2 = 1$

SN = .03 (SNR ≈ 27dB, 27dB)

frac = .6

Covariance Sequence Approximant:

$$\begin{aligned} R_k = {} & (.2486 + j\ .3513_{10}{}^{-2})(1.000 \exp j\ .4717)^{|k|} + c.c. \\ & + (.1003_{10}{}^{-3} - j\ .2234_{10}{}^{-3})(.8381 \exp j\ 2.121)^{|k|} + c.c. \\ & + (.2492 - j\ .3354_{10}{}^{-2})(1.000 \exp -j\ .7848)^{|k|} + c.c. \\ & + (.2633_{10}{}^{-3})(.7696 \exp -j\ 3.142)^{|k|} \\ & + (.1815_{10}{}^{-3})(.1000_{10}{}^{-10})^{|k|} \end{aligned}$$

Both frequency estimates are within .1 percent and their power estimates are within 1 percent. Note that the difference frequency is $\pi/10$ so that with 100 datapoints the error terms due to finite record estimates are vanishingly small. See Figure 4.10B for spectral information plot.

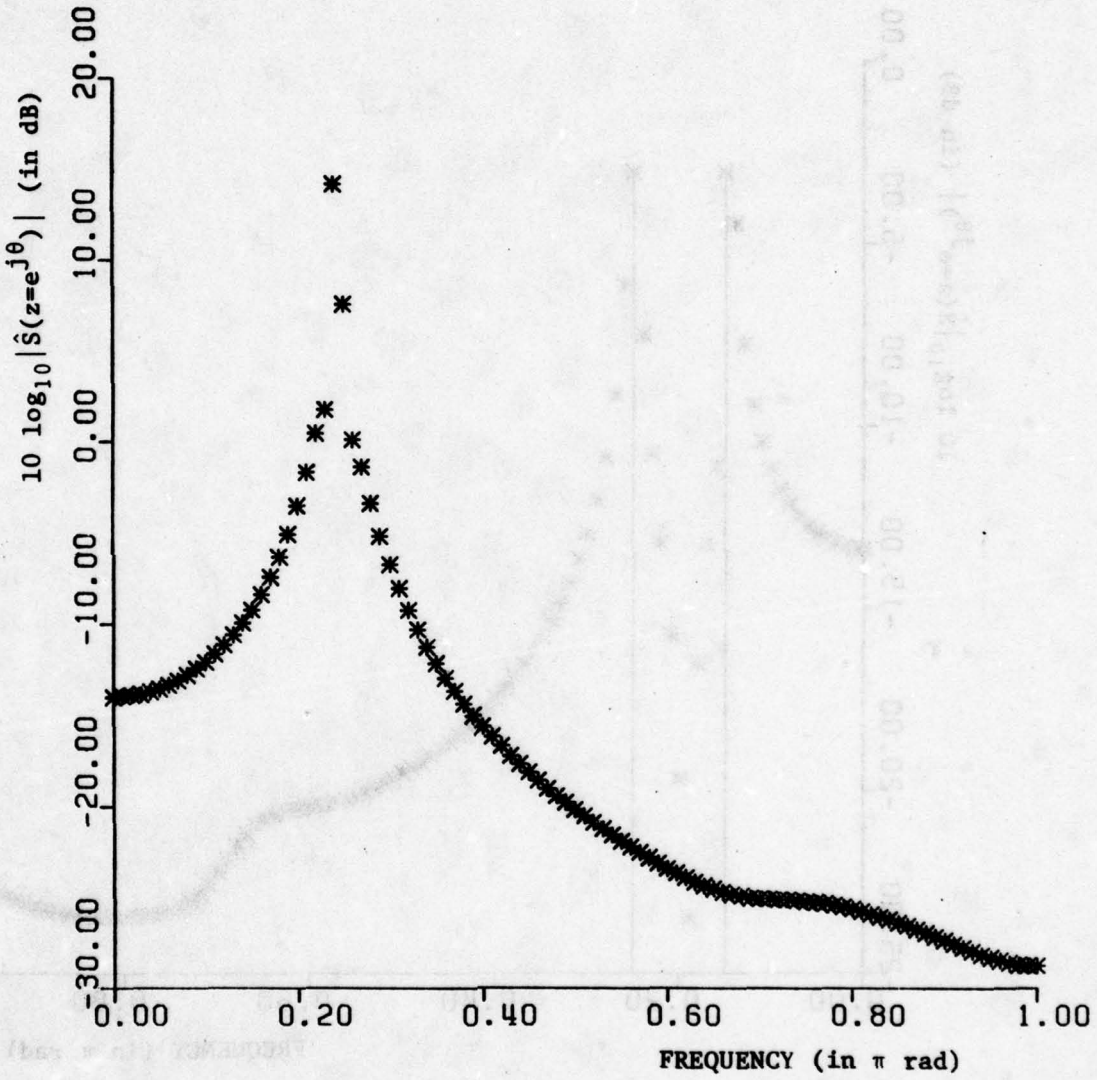

Figure 4.10A Spectral estimate for 2 sines in white noise. SNR = 27 dB, 27 dB. Estimated 20 covariance lags from 100 samples.

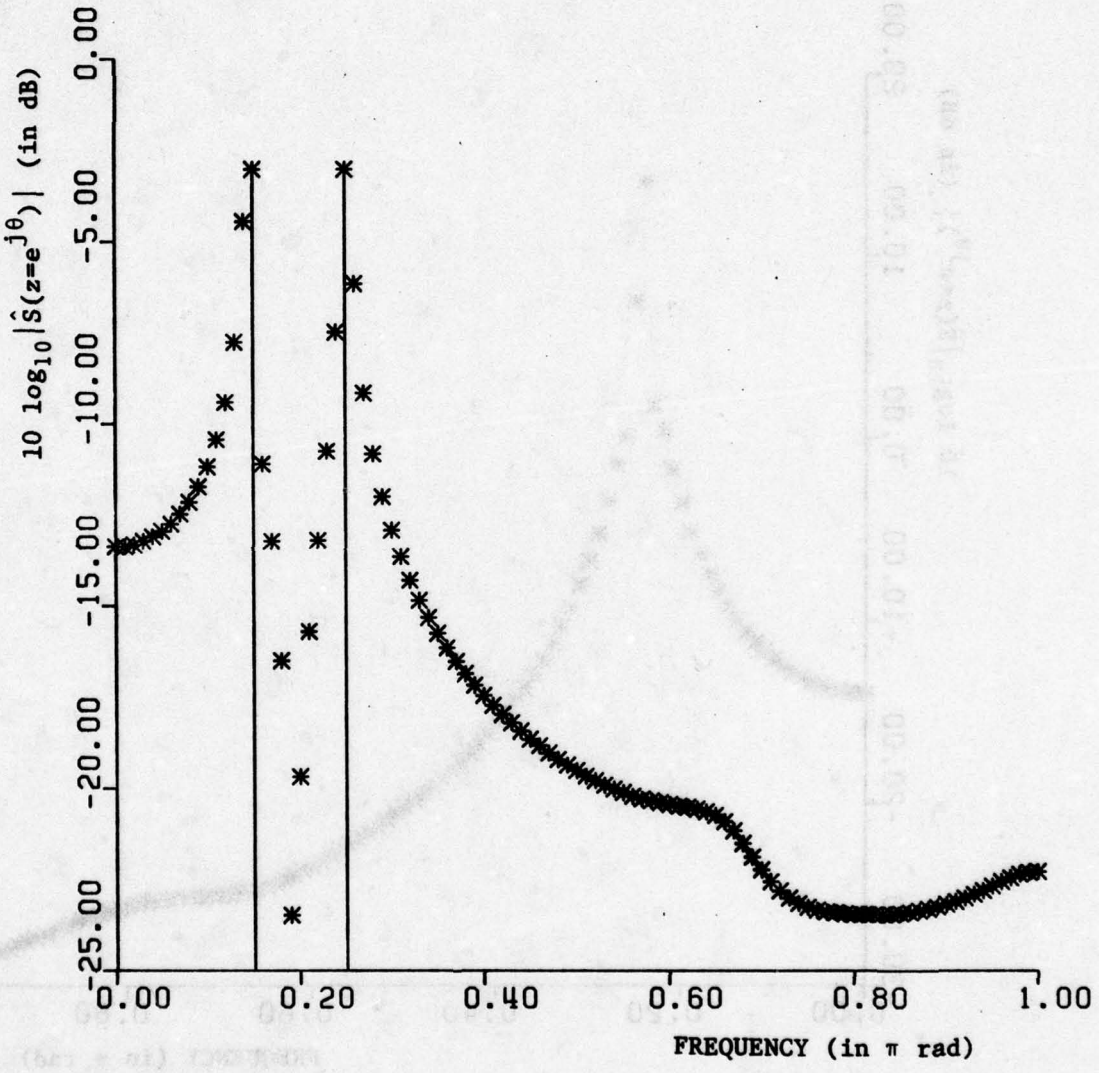

Figure 4.10B Spectral estimate for 2 sines in white noise. SNR = 27 dB, 27 dB. Estimated 20 covariance lags from 100 samples.

C. $S_2 = 1$

SN = .3 (SNR ≈ 7dB, 7dB)

frac = .6

Covariance Sequence Approximant:

$$R_k = (.2589 - j\ .1676_{10}^{-1})(.9940 \exp j\ .4722)^{|k|} + c.c.$$
$$+ (.4916_{10}^{-3})(.7588 \exp j\ 3.142)^{|k|}$$
$$+ (.2442 + j\ .2584_{10}^{-1})(.9997 \exp j\ .7749)^{|k|} + c.c.$$
$$+ (.2625_{10}^{-1} - j\ .5269_{10}^{-3})(.5969 \exp -j\ 2.051)^{|k|} + c.c.$$
$$+ (-.1254_{10}^{-1})(.1000_{10}^{-10})^{|k|}$$

Power and frequency estimates are still within 3 percent and the extra poles seem to represent a noiselevel of -10dB rather well in Figure 4.10C.

D. $S_2 = 1$

SN = 3 (SNR ≈ -13dB, -13dB)

frac = .6

Covariance Sequence Approximant:

$$R_k = (-.2295 - j\ .4725)(.9087 \exp j\ 1.311)^{|k|} + c.c.$$
$$+ (-.8593_{10}^{-2} - j\ .4811)(.8643 \exp -j\ 2.122)^{|k|} + c.c.$$
$$+ (.8077)(1.011)^{|k|}$$
$$+ (.5078_{10}^{1} - j\ .5756_{10}^{1})(.3821 \exp j\ 2.245)^{|k|} + c.c.$$
$$+ (-.6262)(.1000_{10}^{-10})^{|k|}$$

The sinusoidal frequency components are buried in the noise and cannot be recovered even in an averaged frequency component. The stochastic element of the covariance sequence estimate according to (4.39) is now

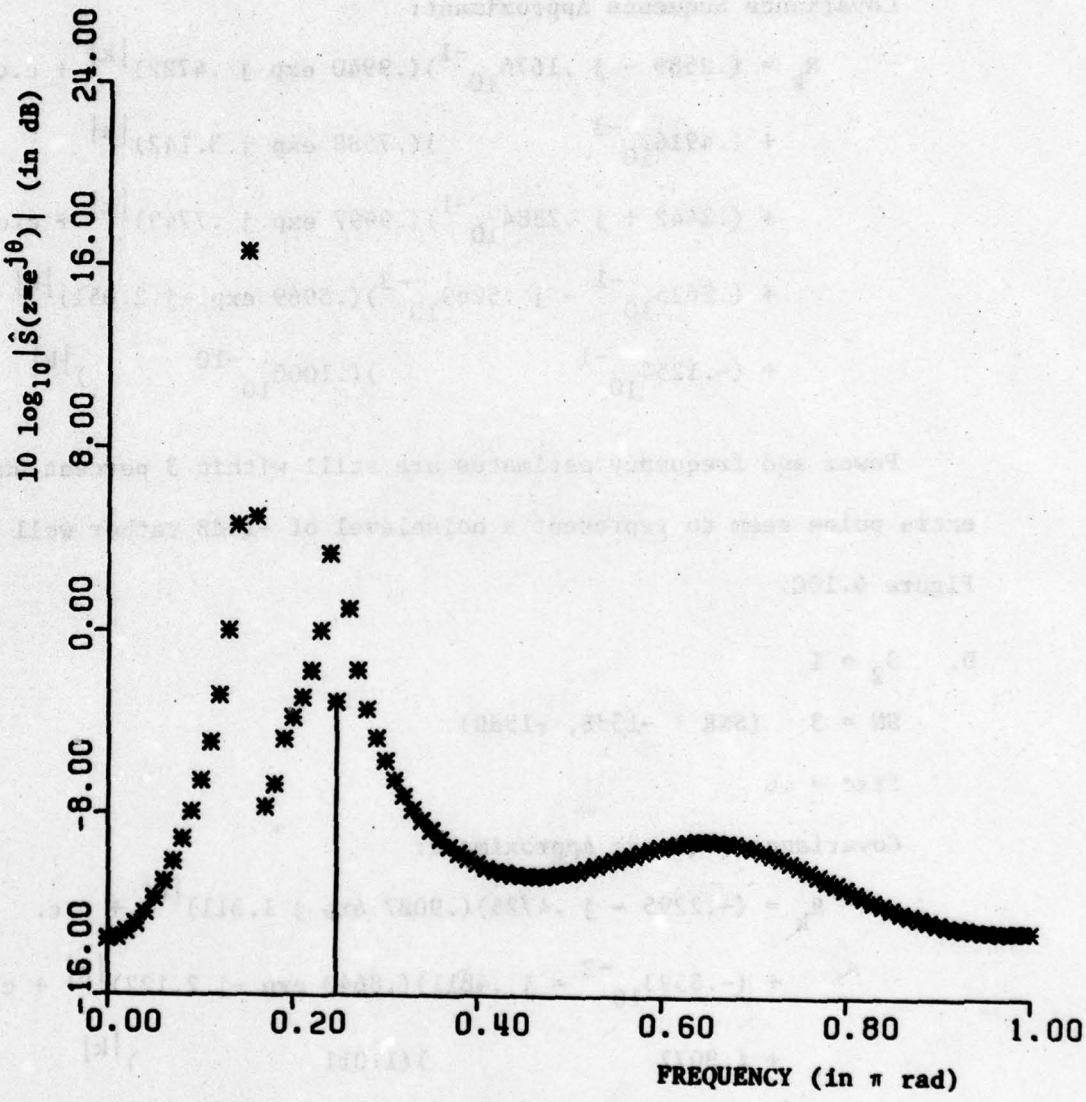

Figure 4.10C Spectral estimate for 2 sines in white noise. SNR = 7 dB, 7 dB. Estimated 20 covariance lags from 100 samples.

of the order 1 which is the maximum value of the covariance sequence for the averaged frequency. Note in the plot of Figure 4.10D the attempt to represent a noise level of 10dB. Note again the unstable pole, so that Figure 4.10D really reflects the spectral density as if the linear weight for the unstable pole were negative and associated with the reflection of the unstable pole.

Finite record based covariance sequence estimates can drastically affect the power estimates corresponding to sinusoidal frequencies. A one-shot procedure is therefore not likely to produce simultaneously good frequency and power estimates when analyzing short data records. Based upon the frequency estimates that result from the first step in the determination of the covariance sequence approximant, the most favorable length of the analysis record may be determined. Calculating the complex linear weight coefficients for the covariance sequence approximant of this thesis from this adjusted analysis record will minimize the error components due to finite length observation records, as in (4.34). When observations are in addition noisy, the variance of the noise covariance estimate may obscure the signal covariance estimate. The result is a loss of resolution. Initially a single frequency component is identified at averaged frequency and power of two separate components. For even noisier data an approximation of the noise process takes place.

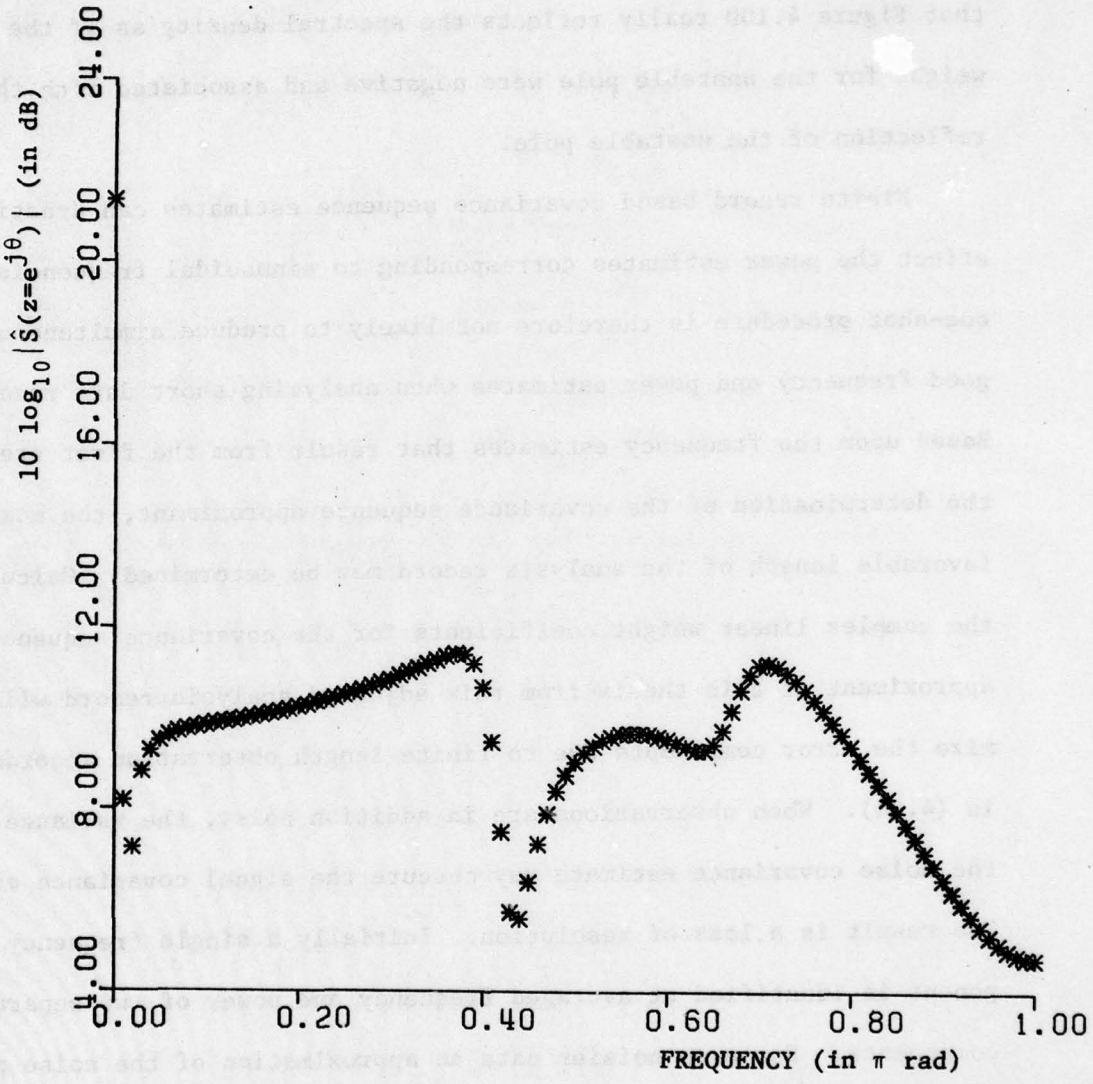

Figure 4.10D Spectral estimate for 2 sines in white noise. SNR = -13 dB, -13 dB. Estimated 20 covariance lags from 100 samples.

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5 COVARIANCE SEQUENCE APPROXIMATION FOR RECURSIVE DIGITAL FILTER DESIGN

In the previous chapter we have seen that the covariance sequence approximant leads to excellent spectral analysis results when it is applied to known covariance sequences. As the model underlying the approximant is an ARMA(M,M) system, it seems natural to use the covariance sequence approximant to design ARMA recursive digital filters, based exclusively on covariance sequence information. In the first section of this chapter we restate a few properties derived earlier and indicate how the possibly negative spectrum estimate associated with the covariance sequence approximant, can be avoided in a slightly modified covariance sequence approximation procedure for the design of recursive digital filters. The resulting filter can be guaranteed to be stable, causal, and have real coefficients.

In Section 5.1 we describe the freedom available in the choice of phase characteristics that can be associated with the resulting magnitude squared approximation. This relative freedom for the phase characteristic is associated with the different possibilities for the numerator polynomial of the filter and several ways are indicated to arrive at a possible set of coefficients. Section 5.3 contains a compilation of numerical considerations associated with the design procedure of this thesis and the final section contains results for several representative low pass recursive digital filter designs.

5.1 Recursive Digital Filter Design

Suppose we wish to design a recursive digital filter H(z) so that its associated spectrum forms a good approximation to a desired spectrum. Many design procedures have been proposed, starting from

the frequency domain [D], [KC], [DU], or from the time domain [BS], [EF]. In general the associated least squares problem is highly nonlinear, which forces upon us a time consuming iterative mode of solution. Simple procedures based on impulse response approximation have generally not succeeded to guarantee stability of the resulting filter [BP], [SH]. To arrive at a more amenable problem statement, modified least squares procedures have been proposed [SM], [MI] and this has recently led to an efficient design procedure for stable ARMA (N,M) digital filters from consistent impulse and covariance sequence data [MR]. Design results with the latter procedure have been reported [SL] and clearly show the influence of the modified least squares error criterion.

We have noted before that an ARMA($M,\bar{M}$) model was the appropriate structure for modeling when it is desired to preserve second order statistical properties in transforming continuous time rational processes to discrete time processes [PS]. An M^{th} order continuous time process leads to an ARMA($M,\bar{M}$) discrete time process, such that

$$C(k) = C_c(\tau=kT) \quad \forall k \qquad 5.1$$

A covariance sequence so derived exactly satisfies an M^{th} order linear homogeneous differential equation with real coefficients. The covariance sequence approximant yields zero covariance sequence prediction error and the results are exact. The covariance sequence approximation procedure then merely serves as an alternative procedure for CI (covariance invariant) digital filter design [PS], [KPSS]. The more difficult task arises when there is no ARMA($M,\bar{M}$) or even ARMA(N,M) system that matches $\{C_k\}_0^{N-1}$ unless N=2M. The latter condition would

constitute a Padé-like approach to covariance sequence approximation and in general would lead to a much higher order for the approximating system than desirable. Also in this case nothing would be known about the match of $\{C_k\}$ for lags larger than N.

We will now indicate how a digital filter with real coefficients can be designed, using the covariance sequence approximant as derived. Let $S_d(\theta)$, $-\pi \leq \theta \leq \pi$, denote an ideal spectrum, with corresponding covariance sequence $\{C_k^d\} \in \ell_2$, to be approximated with an ARMA(M,M^-) digital filter. The least squares approximation problem is to find the parameters $\{A_i, p_i\}_1^M$ for the covariance sequence approximant $\in \ell_1$

$$R_k = \sum_{i=1}^{M} A_i \, p_i^{|k|} \qquad 5.2$$

such that

$$\sum_{-\infty}^{\infty} (R_k - C_k^d)^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |S_d(\theta) - \hat{S}_d(\theta)|^2 \, d\theta \qquad 5.3$$

is minimized. Here $\hat{S}_d(\theta)$ is the spectrum estimate corresponding to the ARMA(M,M^-) covariance sequence approximant. Note the fundamental discrepancy between the ℓ_2 sequence to be approximated and the approximating ℓ_1 sequence. The problem stated above is exactly the general nonlinear problem of (5.2) that we seek to avoid. The modified version of this problem has been outlined in Chapter 3. The main difficulty encountered in the direct application of the modified least squares problem is that $\hat{S}_d(\theta)$ may turn out to be negative for some frequencies, implying that the sequence $\{R_k\}$ is not a covariance sequence and, furthermore, that a factorization of $\hat{S}_d(z)$ to obtain the corresponding filter will give a complex filter. The negativity

of the spectrum estimate is somewhat tolerable in the spectral analysis case, as the problem invariably turns up in the approximation of the low energy parts of the spectrum. In digital filter design it is imperative that a filter with real coefficients result, and when designing ideal filters with very low energy stopbands, zeros on the unit circle cannot be avoided. We note here that in CI design an ARMA(M,M^-) system with real coefficients is known to exist and will be identified, but in the general design problem there is no guarantee that zeros of $\hat{S}_d(z)$ on the unit circle will occur in multiples of two. To avoid the negativity problem we instead approximate the covariance sequence associated with $S_d^{\frac{1}{2}}(\theta)$, which results in a ARMA(M,M^-) covariance sequence approximant with associated spectral density estimate $\hat{S}_d^{\frac{1}{2}}(\theta)$. We now call

$$\hat{S}_d(\theta) = (\hat{S}_d^{\frac{1}{2}}(\theta))^2 \qquad 5.4$$

the approximation to $S_d(\theta)$. Substitution of (4.3a) yields:

$$\hat{S}_d(z) = \frac{N(z)\, N(z^{-1})}{A(z)\, A(z^{-1})\, A(z^{-1})\, A(z)}$$

$$= \frac{|N(z)|^2}{|A(z)|^4} \qquad 5.5$$

Note that $\hat{S}_d(z)$ satisfies all requirements for a proper spectrum. The properties of reality and symmetry are obviously preserved. It has the same poles and zeros as $\hat{S}_d^{\frac{1}{2}}(z)$, but all occur in multiples of two. The important aspect is that zeros of $\hat{S}_d(z)$ on the unit circle now occur in multiples of two so that the coefficients of the associated ARMA($2M,2M^-$) filter are real.

We have seen that the covariance sequence approximation algorithm can be implemented easily to yield the stable denominator polynomial

A(z) of maximum order M for a given covariance sequence. Also the coefficients of A(z) are always real. We note that a change in length of the covariance sequence to be approximated may affect the order as well as the performance of the resulting covariance sequence prediction filter. To preserve stability the obvious choice for the denominator polynomial of the recursive digital filter with spectrum $\hat{S}_d(z)$ will be A(z) A(z). The numerator polynomial can be arranged in many different ways as shown in the following section.

5.2 Numerator Polynomial Coefficients and Phase Characteristics

We are interested in the zeros of $\hat{S}_d(z)$ in (5.5). As they occur in multiples of two it suffices to study the zeros of $\hat{S}_d^{\frac{1}{2}}(z)$. After finding the zeros of $\hat{S}_d^{\frac{1}{2}}(z)$ or equivalently the roots of N(z), the complex roots of $\hat{S}_d(z)$ with radius different from one can be arranged in sets as follows:

$$\{z_o, z_o^*, z_o^{-1}, z_o^{*-1}, z_o, z_o^*, z_o^{-1}, z_o^{*-1}\} \qquad 5.6a$$

Four of these zeros can be chosen for the ARMA($2M,2M^-$) system, provided they occur in complex conjugate pairs, so as to insure the recursive digital filter has real coefficients. Real zeros away from the unit circle occur in following sets:

$$\{\rho_o, \rho_o^{-1}, \rho_o, \rho_o^{-1}\} \qquad 5.6b$$

Two of these roots are to be chosen, with no restrictions. If real roots occur on the unit circle, one of each multiple of two is to be taken. Complex zeros on the unit circle occur in foursomes

$$\{e^{i\theta}, e^{-i\theta}, e^{i\theta}, e^{-i\theta}\} \qquad 5.6c$$

As they again have to be taken in complex conjugate pairs, there is only one way to do so. As a result of the above, the numerator polynomial

of the digital filter has real coefficients. Recall that with a denominator polynomial A(z) A(z) then all digital filter coefficients will be real.

Note that for the zeros away from the unit circle several combinations of zero pairs will be possible and this allows considerable freedom in the phase characteristic to be associated with the ARMA $(2M,2\bar{M})$ filter. It is important to note that this design freedom is implicit, as it was the magnitude squared or spectral performance only that prompted this design procedure. We may therefore achieve approximately linear phase response over a frequency region of importance by cleverly choosing zeros. Alternatively one may achieve partial phase equalization. Furthermore, if the zeros can be chosen to lie inside the unit circle, an invertible digital filter will have been designed. To exercise the available freedom of phase characteristic an additional polynomial rootfinding will have to be executed, even if only on a polynomial of order $\bar{M}$.

A straightforward procedure to arrive at a possible numerator polynomial is to recognize that the following system is causal, has real coefficients and exhibits spectrum $\hat{S}_d(z)$ when driven by white noise:

$$H(z) = \frac{z^{-k} N(z)}{A(z)\ A(z)} \quad ; \quad k \geq \bar{M} \qquad 5.7$$

where again, as in (4.3b):

$$N(z) = \sum_{i=1}^{M} A_i\ (1-p_i)(1+p_i) \prod_{\substack{k=1 \\ \neq i}}^{M} (z-p_k)(z^{-1}-p_k)$$

$$= \sum_{j=-\bar{M}}^{\bar{M}} n_j z^{-j} \quad ; \quad n_j \text{ are real, } n_j = n_{-j} \qquad 5.8$$

The coefficients of $N(z)$ can be determined in straightforward fashion, without polynomial rootfinding or spectral factorization techniques. One may even use to advantage the fact that $N(z)$ is a mirror image polynomial, which cuts the required computational effort roughly in half.

As the latter procedure still may seem to be an exhaustive approach, a more appealing method will be outlined next. Recall

$$\hat{S}_d^{\frac{1}{2}}(z) = \frac{N(z)}{A(z)\ A(z^{-1})}$$

$$= \sum_{i=1}^{M} A_i \frac{(1-p_i)(1+p_i)}{(z-p_i)(z^{-1}-p_i)} \qquad 5.9$$

For any $z = e^{j\theta}$ it is therefore possible to evaluate $N(z = e^{j\theta})$ according to

$$N(z = e^{j\theta}) = A(z)\ A(z^{-1})\ \hat{S}_d^{\frac{1}{2}}(z)\Big|_{z = e^{j\theta}}$$

$$= |A(z)|^2\ \hat{S}_d^{\frac{1}{2}}(z)\Big|_{z = e^{j\theta}} \qquad 5.10$$

Note that all elements of the right hand side of (5.10) are readily available as a_i, p_i and A_i have all been determined. Furthermore $N(z)$ is real and symmetric on $[-\pi, \pi]$, as is easily seen. With (5.8) and (5.10) we may therefore write

$$n_\ell = \text{IDFT}\left\{|A(z)|^2\ \hat{S}_d^{\frac{1}{2}}(z)\Big|_{z = e^{j2\pi k/2M}}\right\} \qquad 5.11$$

The latter approach can also be used to evaluate the coefficients of $N(z)\ N(z^{-1})$ in (5.5). As $N(z)$ represents a pure moving average system with real coefficients, the resulting sequence would represent the covariance sequence associated with $N(z)$. Knowing however a

possible set of coefficients for the numerator polynomial, say from (5.11), the corresponding covariance sequence can be readily calculated.

$$C_N(k) = \begin{cases} \sum_{\ell=-(M-1)}^{M-1} n_\ell \, n_{\ell+k} & ; \quad k = -2(M-1), \ldots, 2(M-1) \\ 0 & \text{otherwise} \end{cases} \qquad 5.12$$

Once the moving average covariance sequence is known, in addition to the order of the generating system, algorithms are available to determine the corresponding minimum phase numerator polynomial for the recursive digital system by factorization techniques [W1].

5.3 Numerical Considerations

In the following section we will design several low pass filters. The starting point is naturally the ideal low pass spectral density, but a modification will be made in that the stopband level is not specified to be zero. Instead the stopband level is thought to represent a hypothetical tolerable noiselevel. We note here that minimization of the covariance sequence prediction error represents a spectral whitening approach. The numerical conditioning of the system of linear equations to be solved for minimization of the covariance sequence prediction error is determined to a large extent by the dynamic range of the desired spectrum [MG]. Specification of a tolerable noiselevel for the stopband performance of the modified ideal filter will therefore stabilize the numerical behavior of the algorithm.

Use of the minimal length covariance sequence to find an approximate ARMA(M,$\bar{M}$) system leads to a Padé-like approximant to the covariance sequence. In the application of complex exponential approximation to one-sided impulse response sequences similar design procedures are known [BP], [SH], [PR], [HI] and several possible problems have been

reported [VB]. Complex roots not occurring in complex conjugate pairs indicate that the rootfinding routine has not converged yet, as the polynomial itself is known to have real coefficients. The obvious solution to such a problem is to allow more iterations in the polynomial routine. In all covariance sequence approximations performed in this thesis the polynomial roots always were either real or occurred in complex conjugate pairs within the specified numerical accuracy.

Partial realization procedures and Padé approximation are known to be very sensitive to noise [VB], [W2]. Most algorithms solving the partial realization problem are for that reason numerically unstable, but numerically stable algorithms have been proposed requiring a slightly increased number of operations [DJ]. In this respect we note that in the covariance sequence approximation procedure of this thesis, equations are always solved in a least squares sense. The order of the approximating system is low compared to the length of the available covariance sequence. This approach has a noise reducing and numerically stabilizing effect. In the proposed digital filter design procedure this sensitivity problem, which would lead to a spectral factorization with complex coefficients, is avoided entirely by design. Noise sensitivity does not play the role it does in partial realization procedures because of the fact that a covariance sequence approximant for the intermediate covariance sequence does not need to have the nonnegative definiteness property. The final result for the spectral approximation is still guaranteed to be nonnegative.

Ideal filters inevitably have discontinuous transitions in the frequency domain, whereas their corresponding covariance sequences will

be relatively smooth and damped. The approximation of such a two-sided covariance sequence with a weighted sum of complex exponentials naturally avoids singular cases of approximating time domain discontinuities which reportedly yields serious problems [MCD]. The modified low pass filters have a square summable covariance sequence whereas our covariance sequence approximant is absolutely summable, which points to a fundamental mismatch between approximated and approximating sequence.

In the approximation of known covariance sequences for both the spectral estimation simulations of Chapter 4 and the digital filter designs of the next section, the denominator polynomials are always minimum phase corresponding to stable digital filters. It is even more noteworthy that these results have been obtained without recursively checking for stability in order to end up with a reduced order but guaranteed stable filter.

5.4 Numerical Results for Recursive Digital Filter Design

Let us assume a normalized sampling frequency of 1 and a normalized cutoff frequency of .1 so that the spectral density specification in Figure 5.1 is relevant. A similar starting point arises in the design of digital filters from a statistical point of view [FS], [SL]. The known covariance sequence associated with the above desired spectral density is derived by sampling the Fourier transform associated with $S_d(f)$:

$$C_k = 2W \operatorname{sinc}(2\pi Wk) + AN \cos\{(1+2W)\tfrac{\pi k}{2}\} (1-2W) \operatorname{sinc}\{(1-2W)\tfrac{\pi k}{2}\} \quad 5.13$$

From Fourier transform properties we may intuitively say that the use of long sequences $\{C_k\}$, will be reflected in emphasis on low frequency approximation. The use of short sequences will correspond to relatively more emphasis on the high frequency approximation error. When minimal

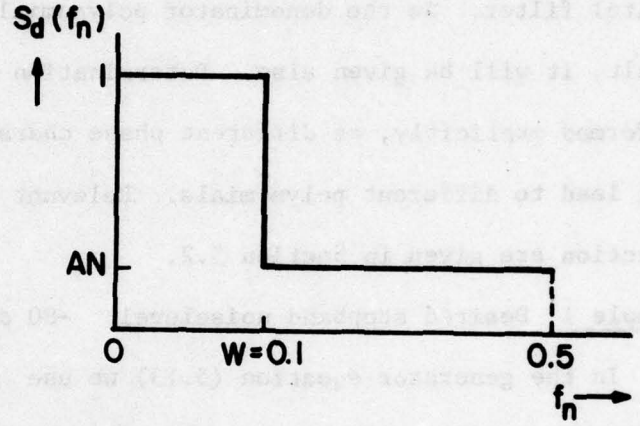

Figure 5.1 Modified lowpass filter specification.

length sequences are used a Padé approximation to the covariance sequence results, with emphasis on the high frequency approximation error and usually characteristic peaking at the cutoff frequency.

For the examples to follow the intermediate covariance sequence approximant will be given, indicating the pole locations for the digital filter. As the denominator polynomial A(z) is an intermediate result, it will be given also. Determination of the zeros was not performed explicitly, as different phase characteristic considerations will lead to different polynomials. Relevant procedures for zero selection are given in Section 5.2.

Example 1 Desired stopband noiselevel: -80 dB

In the generator equation (5.13) we use $AN = 10^{-4}$, corresponding to -40 dB noiselevel for the intermediate spectral density $\hat{S}_d^{\frac{1}{2}}(\cdot)$.

20 covariance sequence lags were used;

Maximum predictor order: M = 10.

Lags 7 through 19 determine the poles.

Lags 0 through 8 determine the zeros.

Covariance Sequence Approximant:

$$R_k = (.8509_{10}{}^{-2} - j\ .8634_{10}{}^{-2})(.9774 \exp j\ .6062)^{|k|} + \text{c.c.}$$

$$+ (.2790_{10}{}^{-1} - j\ .2035_{10}{}^{-1})(.8932 \exp j\ .4986)^{|k|} + \text{c.c.}$$

$$+ (.4239_{10}{}^{-1} - j\ .4219_{10}{}^{-2})(.8087 \exp j\ .2624)^{|k|} + \text{c.c.}$$

$$+ (.4169_{10}{}^{-1} \qquad\qquad)(.8160 \qquad\qquad)^{|k|}$$

$$A(z) = 1 - 5.553z^{-1} + 13.75z^{-2} - 19.60z^{-3} + 17.31z^{-4}$$
$$- 9.455z^{-5} + 2.954z^{-6} - .4067z^{-7}$$

The resulting magnitude squared characteristic for the recursive digital filter of ARMA(14,12) type is given in Figure 5.2. Note here that the specifications practically call for a Padé like approximation. Cutoff frequency peaking is one of the characteristics of such an approximation. Pass band and transition band performance are inferior to stopband performance as might be expected from the use of short covariance sequences. Note also that the maximum tolerable system order was not used because at lower system order the prediction error became negative indicating that the accuracy of the computer was being exceeded.

Example 2 Desired stopband noise level: -60 dB

50 covariance sequence lags were used;

Maximum predictor order: M = 8.

Lags 8 through 49 determine the poles.

Lags 0 through 7 determine the zeros.

Covariance Sequence Approximant:

$$R_k = (.2838_{10}^{-2} - j\ .8560_{10}^{-2})(.9799 \exp j\ .6223)^{|k|} + c.c.$$

$$+ (.1418_{10}^{-1} + j\ .2330_{10}^{-1})(.8881 \exp -j\ .5756)^{|k|} + c.c.$$

$$+ (.4430_{10}^{-1} + j\ .3373_{10}^{-1})(.7356 \exp -j\ .4042)^{|k|} + c.c.$$

$$+ (.7738_{10}^{-1} \qquad\qquad)(.6401 \qquad\qquad)^{|k|}$$

$$+ (.7785_{10}^{-3} \qquad\qquad)(.2267_{10}^{-1} \exp j\ \pi)^{|k|}$$

$$A(z) = 1. - 5.053z^{-1} + 11.56z^{-2} - 15.32z^{-3} + 12.62z^{-4}$$

$$- 6.423z^{-5} + 1.844z^{-6} - .2171z^{-7} - .005948z^{-8}$$

The resulting magnitude squared characteristic for the corresponding ARMA(16,14) digital filter is given in Figure 5.3. The determination

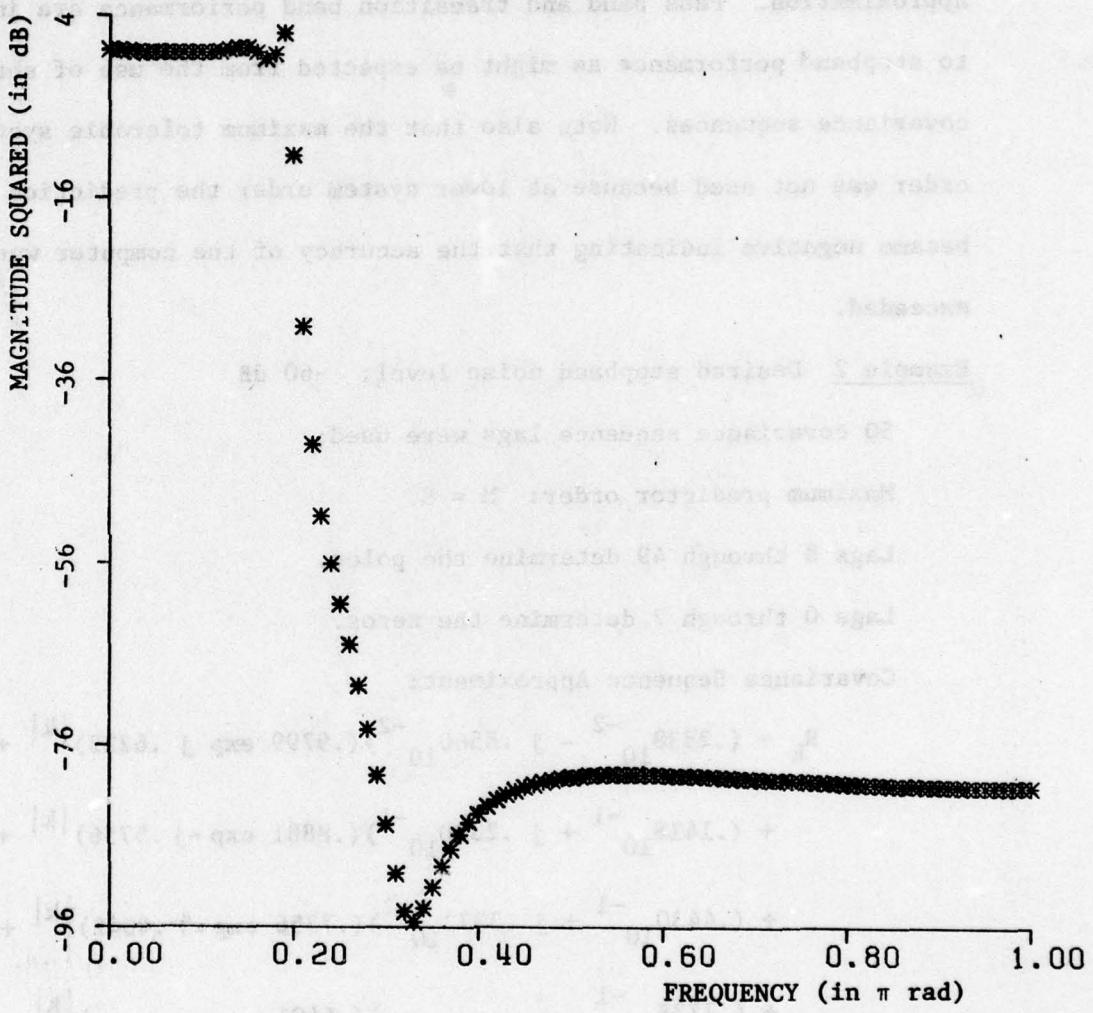

Figure 5.2 Magnitude squared frequency response for ARMA(14,12).

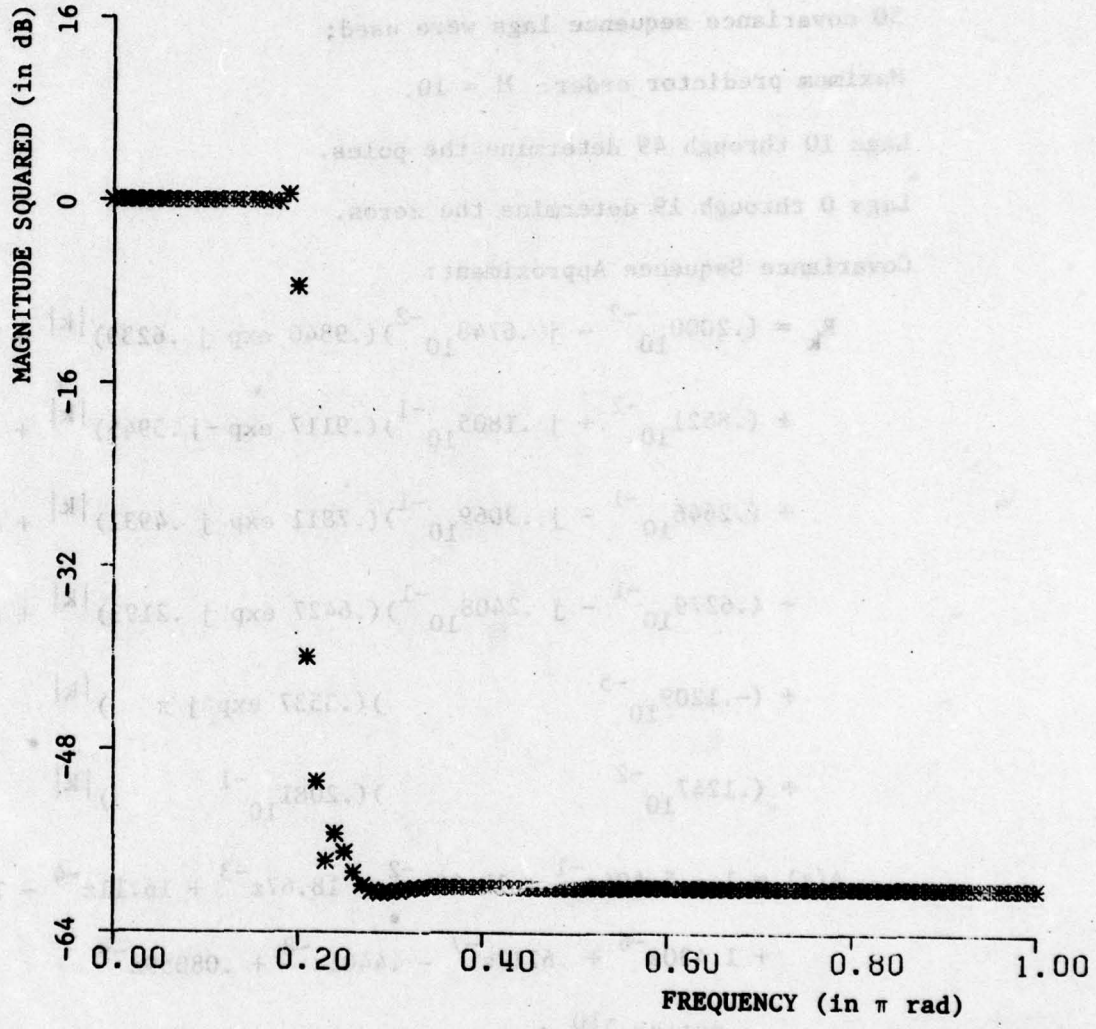

Figure 5.3 Magnitude squared frequency response for ARMA(16,14).

of the zeros is still Padé-like and accounts for the occurrence of slight peaking at the cutoff frequency. Otherwise very reasonable performance is the result in the passband, transition band and stopband.

Example 3 Desired stopband noise level: -60 dB

50 covariance sequence lags were used;

Maximum predictor order: M = 10.

Lags 10 through 49 determine the poles.

Lags 0 through 19 determine the zeros.

Covariance Sequence Approximant:

$$R_k = (.2000_{10}^{-2} - j\ .6748_{10}^{-2})(.9840 \exp j\ .6239)^{|k|} + c.c.$$

$$+ (.8521_{10}^{-2} + j\ .1805_{10}^{-1})(.9117 \exp -j\ .5945)^{|k|} + c.c.$$

$$+ (.2646_{10}^{-1} - j\ .3069_{10}^{-1})(.7811 \exp j\ .4931)^{|k|} + c.c.$$

$$+ (.6279_{10}^{-1} - j\ .2408_{10}^{-1})(.6427 \exp j\ .2193)^{|k|} + c.c.$$

$$+ (-.1209_{10}^{-5})(.3537 \exp j\ \pi)^{|k|}$$

$$+ (.1247_{10}^{-2})(.2081_{10}^{-1})^{|k|}$$

$$A(z) = 1 - 5.406z^{-1} + 13.22z^{-2} - 18.67z^{-3} + 16.11z^{-4} - 7.987z^{-5}$$

$$+ 1.430z^{-6} + .6712z^{-7} - .4401z^{-8} + .08059z^{-9}$$

$$- .001493z^{-10}$$

Figure 5.4 indicates that a very nice result has been achieved in the passband, transitionband and stopband. The use of longer sequences has practically eliminated all ripple in the passband, but simultaneously the desired stopband noise level is achieved. The cutoff rate in the transitionband increases rapidly from about 150 to 600 db/octave.

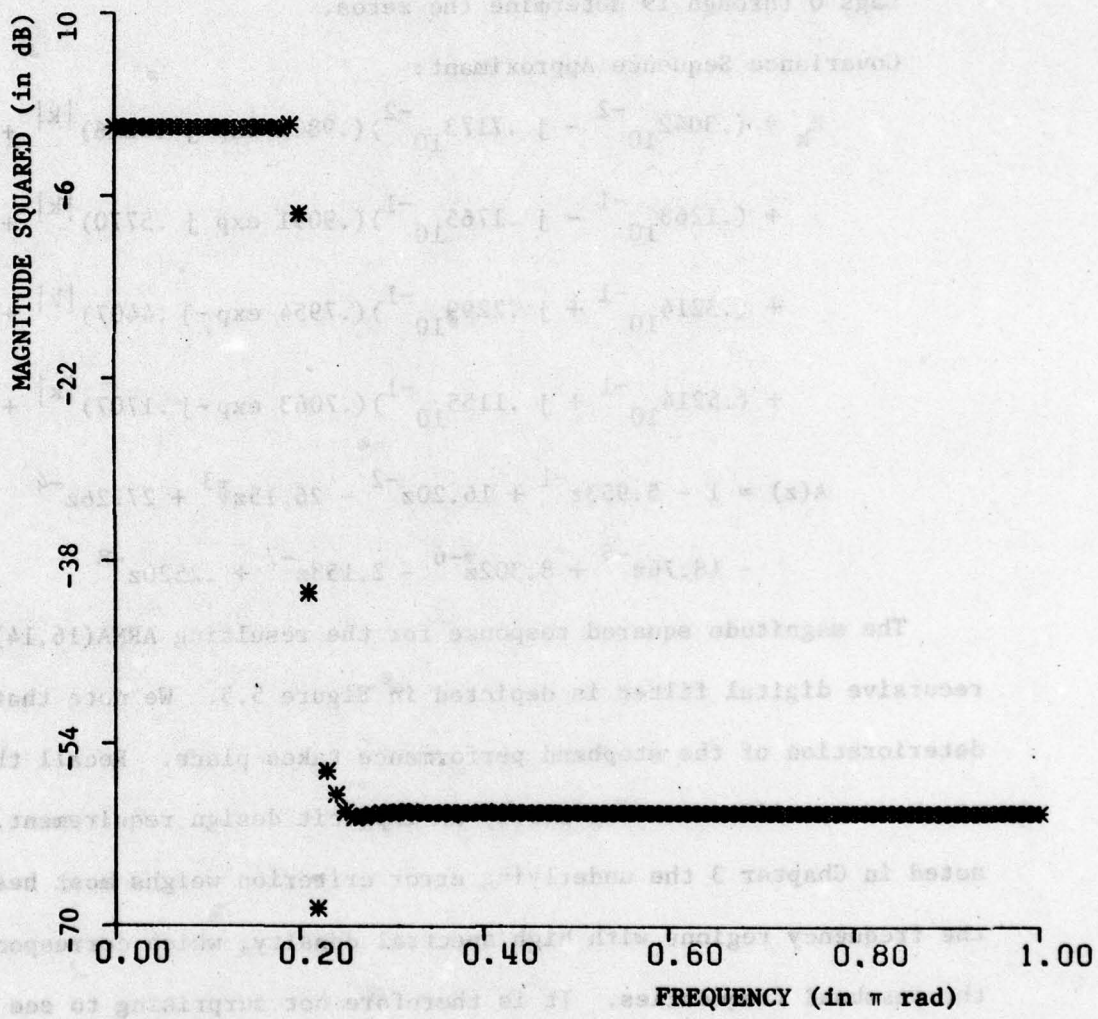

Figure 5.4 Magnitude squared frequency response for ARMA(20,18).

Example 4 Desired stopband noise level: -120 dB

50 covariance sequence lags were used;

Maximum predictor order: M = 8.

Lags 8 through 49 determine the poles.

Lags 0 through 19 determine the zeros.

Covariance Sequence Approximant:

$$R_k = (.3042_{10}^{-2} - j\ .7173_{10}^{-2})(.9829 \exp j\ .6216)^{|k|} + c.c.$$

$$+ (.1268_{10}^{-1} - j\ .1765_{10}^{-1})(.9091 \exp j\ .5770)^{|k|} + c.c.$$

$$+ (.3214_{10}^{-1} + j\ .2299_{10}^{-1})(.7954 \exp -j\ .4407)^{|k|} + c.c.$$

$$+ (.5214_{10}^{-1} + j\ .1155_{10}^{-1})(.7063 \exp -j\ .1707)^{|k|} + c.c.$$

$$A(z) = 1 - 5.953z^{-1} + 16.20z^{-2} - 26.15z^{-3} + 27.26z^{-4}$$

$$- 18.76z^{-5} + 8.302z^{-6} - 2.158z^{-7} + .2520z^{-8}$$

The magnitude squared response for the resulting ARMA(16,14) recursive digital filter is depicted in Figure 5.5. We note that a deterioration of the stopband performance takes place. Recall that the stopband specification was merely an implicit design requirement. As noted in Chapter 3 the underlying error criterion weighs most heavily the frequency regions with high spectral density, which corresponds to the passband frequencies. It is therefore not surprising to see very good passband behavior and deteriorating stopband and transitionband performance. Even so the initial fall off rate is quite high and the stopband rejection eventually reaches -120 dB.

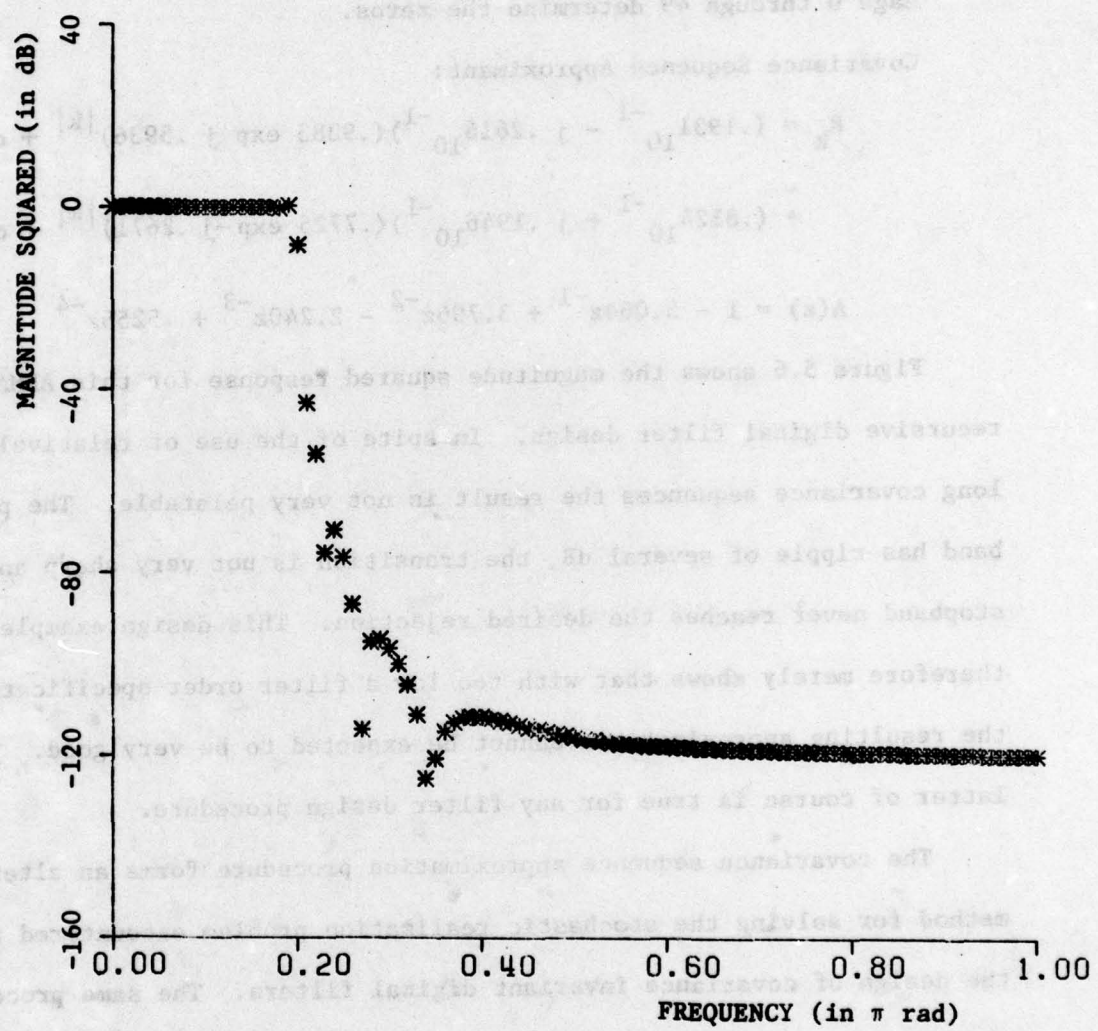

Figure 5.5 Magnitude squared frequency response for ARMA(16,14).

Example 5 Desired stopband noise level: -60 dB

50 covariance sequence lags were used;

Maximum predictor order: M = 4.

Lags 4 through 49 determine the poles.

Lags 0 through 49 determine the zeros.

Covariance Sequence Approximant:

$$R_k = (.1931_{10}^{-1} - j\ .2616_{10}^{-1})(.9383 \exp j\ .5936)^{|k|} + c.c.$$

$$+ (.8124_{10}^{-1} + j\ .1946_{10}^{-1})(.7725 \exp -j\ .2671)^{|k|} + c.c.$$

$$A(z) = 1 - 3.064z^{-1} + 3.796z^{-2} - 2.240z^{-3} + .5255z^{-4}$$

Figure 5.6 shows the magnitude squared response for this ARMA(8,6) recursive digital filter design. In spite of the use of relatively long covariance sequences the result is not very palatable. The passband has ripple of several dB, the transition is not very sharp and the stopband never reaches the desired rejection. This design example therefore merely shows that with too low a filter order specification the resulting approximations cannot be expected to be very good. The latter of course is true for any filter design procedure.

The covariance sequence approximation procedure forms an alternative method for solving the stochastic realization problem encountered in the design of covariance invariant digital filters. The same procedure may be used to find an approximation to an arbitrary covariance sequence for which no exact underlying ARMA(M,N) process exists. The designs resulting from covariance sequence approximation for modified ideal low pass filter designs represent causal, stable ARMA($2M,2M^-$) recursive digital filters with real coefficients. We note that absolutely no

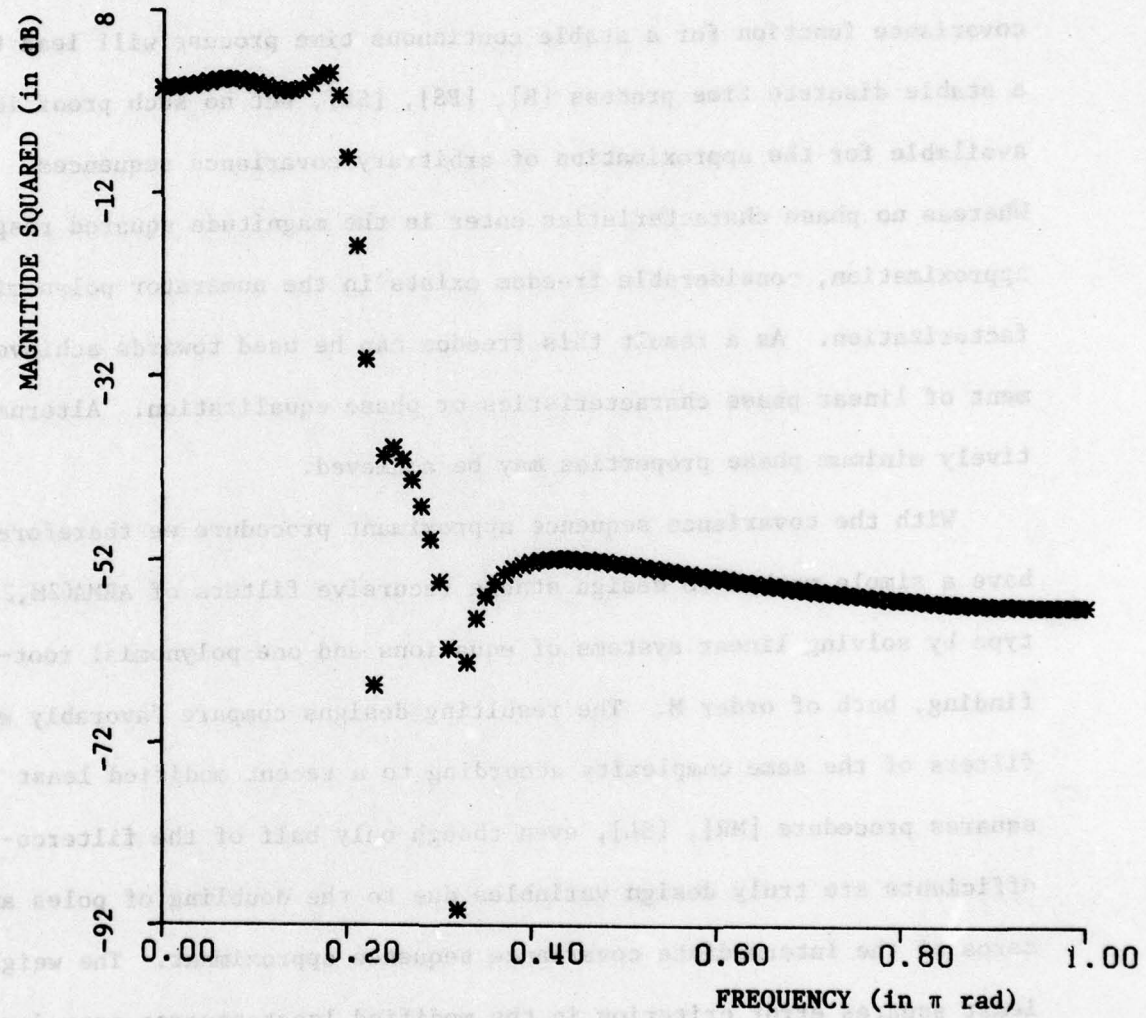

Figure 5.6 Magnitude squared frequency response for ARMA(8,6).

phase information is used in the design. In designs utilizing impulse response sequences at least implicit use of phase information is made.

A stability requirement is easily implementable. In the designs of this thesis an unstable result was never encountered, even without the use of a stability check. For the CI digital filter design the covariance function for a stable continuous time process will lead to a stable discrete time process [B], [PS], [SP], but no such proof is available for the approximation of arbitrary covariance sequences. Whereas no phase characteristics enter in the magnitude squared response approximation, considerable freedom exists in the numerator polynomial factorization. As a result this freedom can be used towards achievement of linear phase characteristics or phase equalization. Alternatively minimum phase properties may be achieved.

With the covariance sequence approximant procedure we therefore have a simple method to design stable recursive filters of ARMA($2M,2M^-$) type by solving linear systems of equations and one polynomial root-finding, both of order M. The resulting designs compare favorably with filters of the same complexity according to a recent modified least squares procedure [MR], [SL], even though only half of the filtercoefficients are truly design variables due to the doubling of poles and zeros of the intermediate covariance sequence approximant. The weighted least squares error criterion in the modified least squares procedure puts heavy emphasis on the stopband performance and obviously at the expense of increased passband ripple.

Each design was performed in single precision on the CYBER 171 at Colorado State University and completed in less than one second of computation time. Due to the fact that a spectral approximation takes

place one can determine the magnitude squared response of the recursive digital filter before actually determining its filter coefficients. In an interactive computation mode this presents the possibility of a visual check on the resulting performance without actually performing the complete design.

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6 CONCLUSIONS

6.1 Summary

In this thesis we have proposed a covariance sequence model consisting of a linear combination of damped complex exponentials. A generalized stochastic process generator has been derived that fits in the framework of the covariance sequence model, as its spectral representation for the covariance sequence is a positive real linear combination of damped complex exponentials. We have thus developed a generalization of the stochastic almost periodic function by allowing the complex exponentials to be damped. A more efficient representation for a large subclass of ARMA(M,M^-) processes is the result.

To identify the parameters in the covariance sequence model a modified least squares approach has resulted in a decoupling of the determination of the complex exponentials and complex linear weights. Only linear systems of equations have to be solved in addition to a polynomial rootfinding. A recursive algorithm is given to solve for the complex exponentials and it is indicated how the stability of the resulting filter can be guaranteed.

When used in a spectral analysis context the covariance sequence approximant of this thesis seems robust against the nonrational Gaussian spectral density. For known covariance sequences with underlying ARMA(M,M^-) systems, the spectral information is obtained without error. When a covariance sequence is estimated from a finite data record the effects of the finite observation length show up in the covariance sequence estimate and consequently in the covariance sequence approximant. These effects are derived and illustrated with realizations from stochastic almost periodic functions where time averaging and ensemble

averaging do not yield the same results for the estimated covariance sequence. Large error terms can arise in the identified complex linear weights. Error terms due to difference frequencies of sinusoidal components can be eliminated by the use of an appropriate length finite observation record, resulting in accurate power and frequency estimates for these signals.

Harmonic processes in white noise form a limiting case for the proposed covariance sequence approximant. Estimated covariance sequences for such processes will yield relevant information concerning the observed process, but for relatively high noise level the signal components may be averaged or not resolved at all.

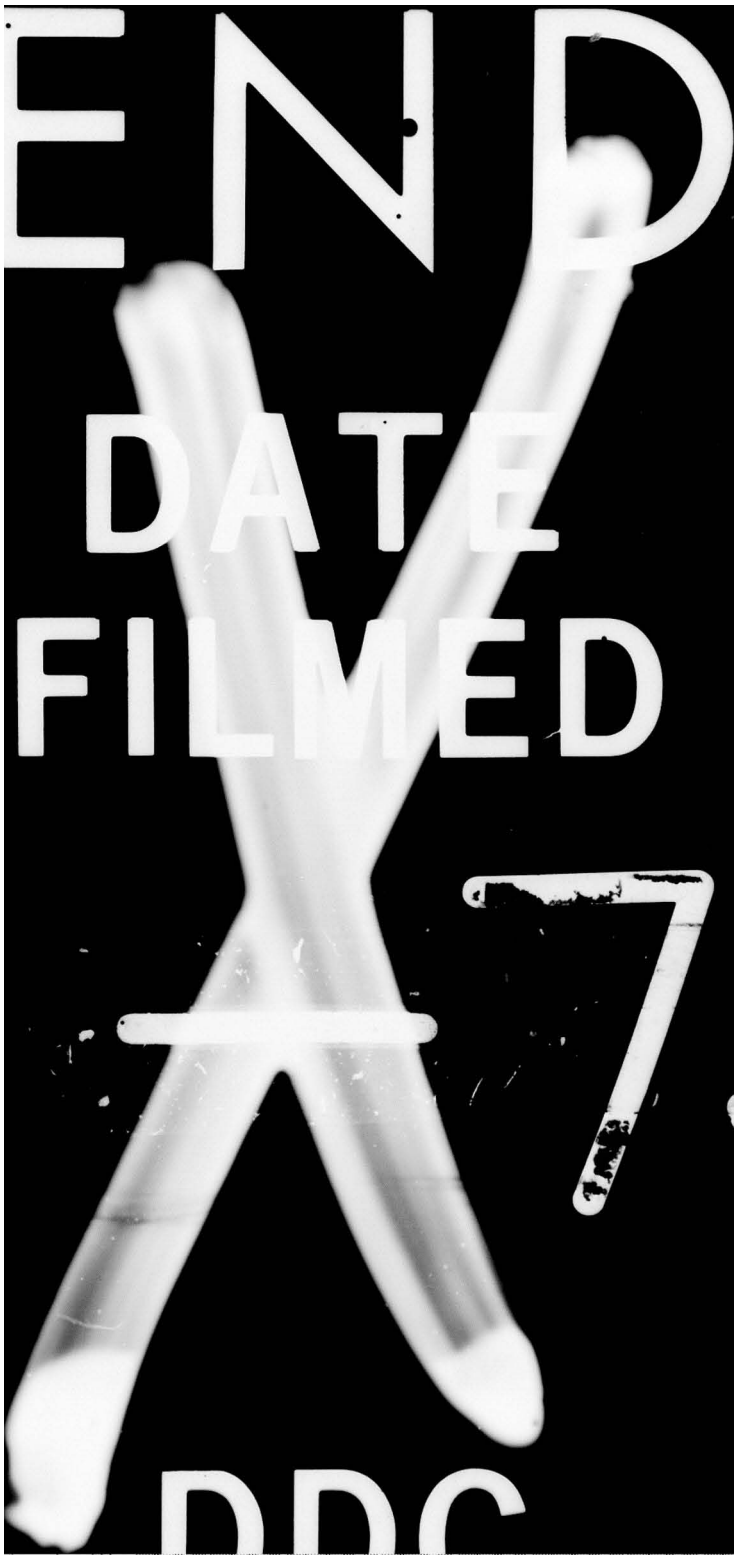
In the application of the covariance sequence approximant to recursive digital filter design from magnitude squared information only, the covariance sequence approximant is sought for the square root of the desired magnitude squared response. This approach avoids by design the problems of an approximating spectrum that is possibly negative for some frequencies. The result is a procedure for the design of causal ARMA($2M,2M^-$) filters with real coefficients of guaranteed stability. This design procedure requires linear systems of equations to be solved in addition to a polynomial rootfinding problem and an inverse DFT. The resulting spectral error criterion for the proposed design procedure is somewhere between modified least squares and least squares. The relative proximity to either is determined by the choice of the system order and the length of the covariance sequences used to determine poles and zeros respectively.

6.2 Recommendations

We have seen that the proposed covariance sequence model is so general that approximating sequences are not necessarily nonnegative definite. Especially for recursive digital filter design this is not tolerable. This prompted the use of a square root spectrum approximation as an intermediate step. Finding general conditions for the nonnegativeness of the spectral estimate could result in a constrained quadratic minimization problem. The solution then would be an ARMA(M,$\bar{M}$) digital filter of half the complexity of the designs produced by Chapter 5. A spectral factorization procedure would be unavoidable however in order to find appropriate numerator polynomial coefficients.

The covariance sequence approximant of this thesis is valid under the assumption of single poles for the underlying system. A different covariance sequence approximant can be defined to incorporate multiple poles. In the solution procedure this would result in derivatives of the independent vectors in the Vandermonde matrix. One can argue that practical systems identified under noisy conditions will hardly ever exhibit multiple poles. The importance of this generalized approach then lies in the fact that multiple poles at zero can be implemented which will allow the identification of ARMA(M,N) systems, where N can be larger than M-1.

While illustrative, the spectrum analysis results of Chapter 4 are not conclusive. Continued investigation should focus on Monte-Carlo studies of spectrum estimates for random amplitude, random phase sinusoids, together with analytical investigation of estimator variances, confidence intervals, and the like.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